

CHAPTER 1

DPP 1.1

Single Correct Answer Type

1. (b) $y = f(x) = \sqrt{(x^2 - 2)^2 + (x - 3)^2} - \sqrt{x^2 + (x^2 - 1)^2}$

Note that the first radical sign describes the distance between $P(x, x^2)$ and $A(3, 2)$ whereas the second radical sign describes the distance between $P(x, x^2)$ and $B(0, 1)$.

Now $PA - PB \leq AB$ for possible positions of P .

Hence $f(x)_{\max} = \text{distance between } AB = \sqrt{9 + 1} = \sqrt{10}$

2. (b) We have $A(\alpha, 6)$, $B(-5, 0)$ and $C(5, 0)$

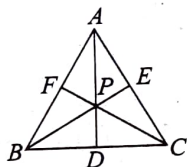
If $AB = AC$, we get $\alpha = 0$.

Hence, we get one value of c

If $AC = BC$, we get $\alpha = -3, 13$

α can take 5 values: 0, 3, -3, 13, -13

3. (d) There exist exactly four triangles ABC , APC , APB and BPC satisfying the given conditions of which three triangles will be obtuse angle.



4. (b) Sides are along lines $y = m_1x$, $y = m_2x$ and $y = 2$

$\therefore$ Vertices of the triangle are $(0, 0)$, $\left(\frac{2}{m_1}, 2\right)$, $\left(\frac{2}{m_2}, 2\right)$

$$\text{Area} = \frac{1}{2} \begin{vmatrix} 0 & 0 \\ \frac{2}{m_1} & 2 \\ \frac{2}{m_2} & 2 \end{vmatrix}$$

$$= 2 \left| \frac{m_2 - m_1}{m_1 m_2} \right|$$

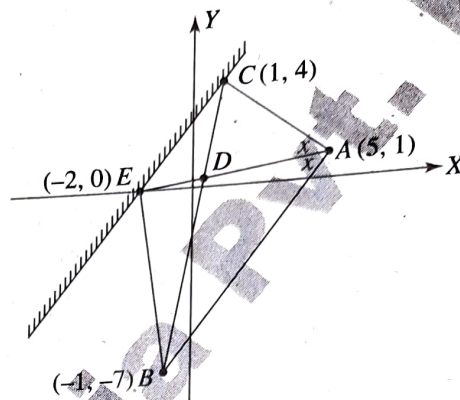
$$\therefore |m_1 - m_2| = \sqrt{(m_1 + m_2)^2 - 4m_1 m_2}$$

$$= \sqrt{(\sqrt{3} + 2)^2 - 4(\sqrt{3} - 1)}$$

$$= \sqrt{11}$$

$$\therefore \text{Area} = 2 \left| \frac{\sqrt{11}}{\sqrt{3} - 1} \right| = \sqrt{33} + \sqrt{11}$$

5. (c)



$$\frac{AD}{DE} = \frac{BD}{DC} = \frac{AB}{AC}$$

$\therefore$ Point $D\left(\frac{1}{3}, \frac{1}{3}\right)$ and $E(-2, 0)$.

Area of $\triangle ADC$ + Area of $\triangle BDE$

$$= \frac{25}{3} + \frac{25}{3}$$

$$= \frac{50}{3} \text{ sq. units}$$

6. (c) Vertices are $A(a, 0)$, $B(-a, 0)$ and $C(b, c)$

$\therefore$ Centroid is $G\left(\frac{b}{3}, \frac{c}{3}\right)$

$$\frac{AB^2 + BC^2 + CA^2}{GA^2 + GB^2 + GC^2}$$

$$= \frac{4a^2 + (a+b)^2 + c^2 + (a-b)^2 + c^2}{\left(\frac{b}{3} - a\right)^2 + \left(\frac{c}{3}\right)^2 + \left(\frac{b}{3} + a\right)^2 + \left(\frac{c}{3}\right)^2 + \left(\frac{2b}{3}\right)^2 + \left(\frac{2c}{3}\right)^2}$$

$$= \frac{4a^2 + 2c^2 + 2a^2 + 2b^2}{\frac{2b^2}{9} + 2a^2 + \frac{6c^2}{9} + \frac{4b^2}{9}} = 3$$

7. (b) Let internal bisector of $\angle APB$ meets the side AB at C .

We know that $\frac{AC}{BC} = \frac{AP}{PB}$

$$\therefore \frac{AC}{BC} = \frac{3}{2}$$

$\therefore$ coordinates of C are $\left(\frac{3(10) + 2(5)}{3 + 2}, \frac{3(12) + 2(2)}{3 + 2}\right)$ or $(8, 8)$

Hence the internal bisector of $\angle APB$ always passes through point $(8, 8)$

8. (d) Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ and $P(h, k)$ be the points.

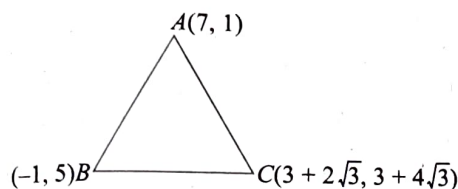
Now, $a AP^2 + b BP^2 + c CP^2$

$$\begin{aligned}
 &= a[(h-x_1)^2 + (k-y_1)^2] + b[(h-x_2)^2 + (k-y_2)^2] \\
 &\quad + c[(h-x_3)^2 + (k-y_3)^2] \\
 &= [h^2(a+b+c) - 2h(ax_1+bx_2+cx_3) + (ax_1^2+bx_2^2+cx_3^2)] \\
 &\quad + [k^2(a+b+c) - 2k(ay_1+by_2+cy_3) + (ay_1^2+by_2^2+cy_3^2)]
 \end{aligned}$$

$$\text{which is minimum when } h = \frac{ax_1+bx_2+cx_3}{a+b+c}, k = \frac{ay_1+by_2+cy_3}{a+b+c}$$

So, P is incentre of $\triangle ABC$.

9. (a)



$$\therefore AB = BC = CA = 4\sqrt{5}$$

i.e., given triangle is equilateral.

$$\begin{aligned}
 \text{Hence, incentre} &= \left(\frac{7-1+3+2\sqrt{3}}{3}, \frac{1+5+3+4\sqrt{3}}{3} \right) \text{ (centroid)} \\
 &= \left(3 + \frac{2}{\sqrt{3}}, 3 + \frac{4}{\sqrt{3}} \right)
 \end{aligned}$$

10. (a) Clearly $OP = OQ = OR$, where O is orthocenter

$\therefore O(0,0)$ is the circum centre of $\triangle PQR$.

Also, given orthocenter = $(0, 0)$

Thus, orthocentre and circumcentre coincide,

So, triangle is equilateral

$\therefore$ Centroid of $\triangle PQR = (0, 0)$

$\Rightarrow \cos \alpha + \cos \beta + \cos \gamma = 0$, and $\sin \alpha + \sin \beta + \sin \gamma = 0$

Squaring and adding we get

$$\cos(\alpha - \beta) + \cos(\beta - \gamma) + \cos(\gamma - \alpha) = -\frac{3}{2}$$

11. (b) Centroid of the given triangle = $(4, 2)$

So, centroid of the image triangle is itself the image of the original centroid.

$\therefore G_1 \equiv (4, -2), G_2 \equiv (-4, 2), G_3 \equiv (2, 4)$

$$\therefore \text{Area of } \triangle G_1 G_2 G_3 \text{ is } \frac{1}{2} \begin{vmatrix} 4 & -2 \\ -4 & 2 \\ 2 & 4 \end{vmatrix} = 20 \text{ sq. units}$$

Comprehension Type

12. (a) Let $BP : PC = \alpha : 1$

$$\therefore P \left(\frac{\alpha x_3 + x_2}{\alpha + 1}, \frac{\alpha y_3 + y_2}{\alpha + 1} \right)$$

P lies on $lx + my + n = 0$

$$\therefore l \left(\frac{\alpha x_3 + x_2}{\alpha + 1} \right) + m \left(\frac{\alpha y_3 + y_2}{\alpha + 1} \right) + n = 0$$

$$\therefore \alpha = \frac{-(lx_2 + my_2 + n)}{(lx_3 + my_3 + n)}$$

Similarly, if $CQ : QA = \beta : 1$

$$\therefore \beta = \frac{-(lx_3 + my_3 + n)}{(lx_1 + my_1 + n)}$$

and if $AR : RB = \gamma : 1$

$$\therefore \gamma = \frac{-(lx_1 + my_1 + n)}{(lx_2 + my_2 + n)}$$

$$\therefore \frac{BP}{PC} \cdot \frac{CQ}{QA} \cdot \frac{AR}{RB} = -1$$

13. (d) From the above result $\Rightarrow \frac{2}{1} \cdot \frac{1}{3} \cdot \frac{AR}{RB} = -1$

$$\therefore \frac{AR}{RB} = -\frac{3}{2}$$

$\therefore R$ divides AB $3 : 2$ externally.

14. (c) Since $\cos A = \frac{AB^2 + AC^2 - BC^2}{2 \cdot AB \cdot AC}$

A is obtuse

$\therefore \cos A < 0$

$\therefore AB^2 + AC^2 < BC^2$

Also A, B, C are non-collinear.

$$\Rightarrow 10 > \beta^2 + 4 + 1 + (\beta - 1)^2$$

$$\Rightarrow (\beta - 2)(\beta + 1) < 0$$

$$\Rightarrow \beta \in (-1, 2)$$

But A, B, C are collinear for $\beta = \frac{2}{3}$

$\Rightarrow$ Correct interval for B is

$$\left(-1, \frac{2}{3} \right) \cup \left(\frac{2}{3}, 2 \right)$$

15. (c) Whenever A is obtuse it is the largest angle also. But there may be more values of β for which A is largest but is not obtuse. Angle A will be largest if $BC > AC, BC > AB, A, B, C$ are not collinear

$$\Rightarrow 10 > 1 + (\beta - 1)^2, 10 > 4 + \beta^2, \beta \neq \frac{2}{3}$$

$$\Rightarrow \beta^2 - 2\beta - 8 < 0, \beta^2 < 6$$

$$\Rightarrow -2 < \beta < 4, -\sqrt{6} < \beta < \sqrt{6}$$

$$\beta \in \left(-2, \frac{2}{3} \right) \cup \left(\frac{2}{3}, \sqrt{6} \right)$$

DPP 1.2

Single Correct Answer Type

1. (b) $\cot A + \cot B = \text{constant}$

$$\Rightarrow \frac{AD}{CD} + \frac{BD}{CD} = \text{constant, (where } D \text{ is the foot of perpendicular from } C \text{ to } AB)$$

$$\Rightarrow \frac{AD + BD}{CD} = \text{constant}$$

$$\Rightarrow \frac{2a}{CD} = \text{constant}$$

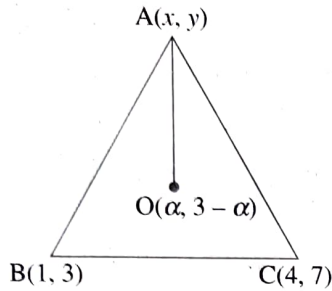
S.4 Coordinate Geometry

$$\Rightarrow CD = \text{constant}$$

C is moving at a constant distance from AB

$\therefore$ locus of C is a line parallel to AB

2. (c)



Orthocenter lies on the line $x + y = 3$

Let orthocenter be $O(\alpha, 3 - \alpha)$

$$BO \perp AC$$

$$\Rightarrow \frac{y-7}{x-4} \cdot \frac{\alpha}{1-\alpha} = -1$$

$$\Rightarrow \frac{\alpha-1}{\alpha} = \frac{y-7}{x-4}$$

$$\Rightarrow 1 - \frac{1}{\alpha} = \frac{y-7}{x-4}$$

$$\Rightarrow \frac{1}{\alpha} = 1 - \frac{y-7}{x-4}$$

$$\Rightarrow \alpha = \frac{x-4}{x-y+3}$$

Also, $AO \perp BC$

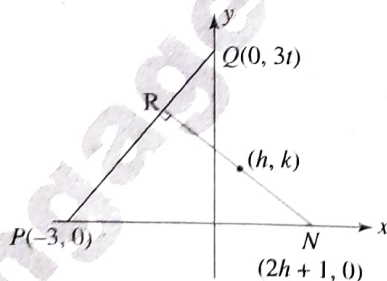
$$\Rightarrow \frac{y-3+\alpha}{x-\alpha} \times \frac{4}{3} = -1$$

$$\Rightarrow 4y - 12 + 4\alpha = -3x + 3\alpha$$

$$\Rightarrow \alpha = -3x - 4y + 12$$

$$\therefore \text{The locus is } \frac{x-4}{x-y+3} = -3x - 4y + 12$$

3. (d)



$$P(-3, 0), Q(0, 3t), R(-1, 2t)$$

Let the mid point of RN be (h, k) .

$$\therefore k = t.$$

$$RN \perp PQ \Rightarrow \frac{2t}{-2h-2} \times \frac{3t}{3} = -1$$

$$\Rightarrow 2t^2 = 2h + 2$$

$$\Rightarrow t^2 = h + 1$$

$$\Rightarrow k^2 = h + 1$$

Locus of (h, k) is $y^2 = x + 1$

4. (a) Let (h, k) be point of intersection then

$$\frac{h}{a} + \frac{k}{b} = 1$$

(i)

(ii)

$$\text{and } ah + bk = 1$$

Also it is given that $a^2 + b^2 = 1$

Multiplying (i) and (ii), we get $h^2 + k^2 + hk \left(\frac{b}{a} + \frac{a}{b} \right) = 1$

$$\text{or } h^2 + k^2 + hk = 1$$

$$\text{or } x^2 + y^2 + xy - 1 = 0$$

5. (a) Rectangle ABCD has vertices $A(0, 0)$ and $D(4, 4)$.

Let B is (h, k) .

$$AB \perp BC$$

$$\therefore \frac{k}{h} \cdot \frac{k-4}{h-4} = -1$$

$\therefore$ The locus of the exterminates of the other diagonal is

$$(x-0)(x-4) + (y-4)(y-0) = 0$$

$$\text{or } x^2 + y^2 - 4x - 4y = 0$$

6. (c) All the altitudes pass through $(0, 0)$. Hence, origin is the orthocentre $A \equiv (t, -t)$.

Let $B \equiv (4t_1, t_1)$ and $C \equiv (t_2, 2t_2)$ satisfying BE and CF.

$$m_{BE} = \frac{1}{4}, m_{CF} = 2, m_{AC} = \frac{2t_2+t}{t_2-t} \text{ and } m_{AB} = \frac{t_1+t}{4t_1-t}$$

$$\text{So } \frac{1}{4} \left(\frac{2t_2+t}{t_2-t} \right) = -1 \text{ and } 2 \left(\frac{t_1+t}{4t_1-t} \right) = -1$$

$$\Rightarrow t_2 = \frac{t}{2} \text{ and } t_1 = -\frac{t}{6}$$

$$\text{So } C \equiv \left(\frac{t}{2}, t \right) \text{ and } B \equiv \left(-\frac{2}{3}t, -\frac{t}{6} \right)$$

Let $G(x_1, y_1)$ be centroid of $\triangle ABC$ and 't' varies.

$$\text{So } x_1 = \frac{1}{3} \left(t - \frac{2}{3}t + \frac{t}{2} \right) = \frac{5t}{18} \text{ and}$$

$$y_1 = \frac{1}{3} \left(-t - \frac{t}{6} + t \right) = -\frac{t}{18}$$

Eliminating 't', we get the locus as $-x_1 = 5y_1$ or $x = -5y$.

7. (d) A line parallel to y-axis has the slope ∞ .

$$\therefore (a^2 - 1)m^2 - (2a - 3)m + a = 0 \text{ must have one root as infinity.}$$

$\therefore a^2 - 1 = 0$ ($\because$ both roots cannot be ∞ , otherwise $a^2 - 1 = 0$, $2a - 3 = 0$ are to hold for the same a)

8. (d) $m_1 = 3, m_2 = \frac{1}{2}$, and $m_3 = m$

Let the angle between first and third line is θ_1 and between second and third is θ_2 . Then

$$\tan \theta_1 = \frac{3-m}{1+3m} \text{ and } \tan \theta_2 = \frac{m-\frac{1}{2}}{1+\frac{m}{2}}$$

$$\text{But } \theta_1 = \theta_2 \Rightarrow \frac{3-m}{1+3m} = \frac{m-\frac{1}{2}}{1+\frac{m}{2}}$$

$$\Rightarrow 7m^2 - 2m - 7 = 0 \Rightarrow m = \frac{1 \pm 5\sqrt{2}}{7}$$

9. (c) Let $B(a, b)$, $C(c, b)$, $A(a, d)$.

Then D (mid point of BC) is $\left(\frac{a+c}{2}, b\right)$

E (mid point of AB) is $\left(a, \frac{b+d}{2}\right)$

$$\text{Given slope of } CE = 1 \Rightarrow \frac{b - \frac{b+d}{2}}{c-a} = 1 \Rightarrow \frac{(b-d)}{c-a} = 2$$

$$\text{Slope of } AD = \frac{\frac{b-d}{2} - d}{a+c-a} = 2 \frac{(b-d)}{c-a} = 4$$

10. (a) Slope of line = $\frac{\sqrt{3}-3}{3-\sqrt{5}}$

Clearly numerator and denominator have different irrational number.

So, there will be no rational point on the line.

11. (c) $\left(-2, \frac{2\pi}{3}\right)$

$$x = r \cos \theta \text{ and } y = r \sin \theta$$

$$\therefore x = (-2) \cos \frac{2\pi}{3} \text{ and } y = (-2) \sin \frac{2\pi}{3}$$

$$\therefore x = (-2) \left(-\frac{1}{2}\right) \text{ and } y = (-2) \left(\frac{\sqrt{3}}{2}\right)$$

$$\therefore (1, -\sqrt{3})$$

12. (a) Perpendicular to $\sqrt{3} \sin \theta + 2 \cos \theta = \frac{4}{r}$ is

$$\sqrt{3} \sin \left(\frac{\pi}{2} + \theta\right) + 2 \cos \left(\frac{\pi}{2} + \theta\right) = \frac{k}{r}$$

It is passing through $(-1, \pi/2)$

$$\therefore \sqrt{3} \sin \pi + 2 \cos \pi = \frac{k}{-1} \Rightarrow k = 2$$

$$\therefore \sqrt{3} \cos \theta - 2 \sin \theta = \frac{2}{r} \Rightarrow 2 = \sqrt{3} r \cos \theta - 2r \sin \theta = 2.$$

13. (c) Transformed equation

$$(x+2)^2 + 2(y-3) - 3 = 0$$

$$\Rightarrow x^2 + 4x + 2y - 5 = 0$$

Multiple Correct Answers Type

14. (b, d)

Put $x = r \cos \theta$, $y = r \sin \theta$ in given equation

$$5r^2 - 3r^2 \sin 2\theta = 4$$

$$\Rightarrow r^2 = \frac{4}{5 - 3 \sin 2\theta}$$

$$\Rightarrow r_{\min}^2 = 4/8 = 1/2 \text{ (when } \sin 2\theta = -1)$$

$$\Rightarrow r_{\min} = \frac{1}{\sqrt{2}} \text{ at } 2\theta = \frac{3\pi}{2}, \frac{7\pi}{2}; \text{ as } [2\theta \in (0, 4\pi)]$$

$$\text{So } \theta = \frac{3\pi}{4}, \frac{7\pi}{4}$$

$$\text{So points are } \left(\frac{1}{\sqrt{2}} \cos \theta, \frac{1}{\sqrt{2}} \sin \theta\right)$$

$$= \left(-\frac{1}{2}, \frac{1}{2}\right) \text{ and } \left(\frac{1}{2}, -\frac{1}{2}\right)$$

15. (a, c)

$$x \cos \alpha + y \sin \alpha = P$$

Axis rotated through angle θ .

Transformed equation

$$\cos \alpha (x \cos \theta - y \sin \theta) + \sin \alpha (x \sin \theta + y \cos \theta) = P$$

$$x \cos(\alpha - \theta) + y \sin(\alpha - \theta) = P \Rightarrow x \cos \beta + y \sin \beta = P$$

where,

$$\cos \beta = \cos(\alpha - \theta), \sin \beta = \sin(\alpha - \theta)$$

CHAPTER 2

DPP 2.1

Single Correct Answer Type

1. (b) The line $\frac{x}{p} + \frac{y}{q} = 1$, where 'p' is a prime and 'q' is a positive integer, passes through (4, 3).

$$\therefore \frac{4}{p} + \frac{3}{q} = 1$$

$$\therefore q = \frac{3p}{p-4} = 3 + \frac{12}{p-4}$$

Since, q is a positive integer, $p > 4$ and $(p-4)$ divides 12, there are only two such prime numbers,

$$p = 5 \text{ and } p = 7$$

2. (c) Equation of the line in normal form is $x \cos \alpha + y \sin \alpha = p$;
 $\therefore l = \cos \alpha, m = \sin \alpha, n = -p$
 $\therefore l^2 + m^2 = 1$ and $n < 0$

3. (b) $\angle ABC = \tan \theta = \frac{\frac{1}{2} - 1}{1 + \frac{1}{2}} = -\frac{1}{3}$

$$\left(m_1 = \frac{1}{2}, m_2 = 1 \right)$$

$$\Rightarrow B \equiv \left(\frac{1}{13}, -\frac{1}{13} \right)$$

8. (c) Solving $y = m_r x$ and $x + y = 1$, we get $x = \frac{1}{1+m_r}$ and $y = \frac{m_r}{1+m_r}$.

Thus the points of intersection of the three lines on the transversal are

$$P \left(\frac{1}{1+m_1}, \frac{m_1}{1+m_1} \right), Q \left(\frac{1}{1+m_2}, \frac{m_2}{1+m_2} \right) \text{ and } R \left(\frac{1}{1+m_3}, \frac{m_3}{1+m_3} \right)$$

According to question

$$PQ = QR$$

$$\begin{aligned} & \left(\frac{1}{1+m_1} - \frac{1}{1+m_2} \right)^2 + \left(\frac{m_1}{1+m_1} - \frac{m_2}{1+m_2} \right)^2 \\ &= \left(\frac{1}{1+m_2} - \frac{1}{1+m_3} \right)^2 + \left(\frac{m_2}{1+m_2} - \frac{m_3}{1+m_3} \right)^2 \end{aligned}$$

$$\Rightarrow \frac{m_2 - m_1}{1+m_1} = \frac{m_3 - m_2}{1+m_3}$$

$$\Rightarrow \frac{1+m_2}{1+m_1} - 1 = 1 - \frac{1+m_2}{1+m_3}$$

$$\Rightarrow \frac{1+m_2}{1+m_1} + \frac{1+m_2}{1+m_3} = 2$$

$$\Rightarrow 1+m_2 = \frac{2(1+m_1)(1+m_3)}{(1+m_1) + (1+m_3)}$$

$$\Rightarrow 1+m_1, 1+m_2, 1+m_3 \text{ are in H.P.}$$

Hence, the lines which make the triangle are $x - y = 2$, $x = 3$ and $y = 4$.

The intersection points of these lines are (6, 4), (3, 1) and (3, 4)

$$\therefore \Delta = \frac{1}{2} |6(-3) + 3(0) + 3(3)| = \frac{9}{2}$$

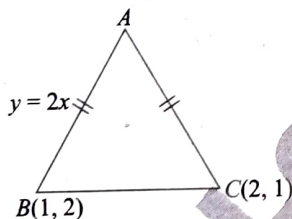
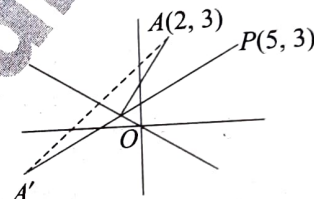
6. (c) Any point (x_1, y_1) after shifting 2 units along + y axis becomes $(x_1, y_1 + 2)$ and after shifting along + x axis it will be $(x_1 + 1, y_1 + 2)$, which satisfy the same equation.

7. (c) Let A' is image of $A(2, 3)$ on $x + y = 0$

$$\Rightarrow A' = (-3, -2)$$

Now, B is the point of intersection of PA' with $x + y = 0$

Equation of line PA' is $y - 3 = (5/8)(x - 5)$ or $5x - 8y - 1 = 0$



Hence, $-\frac{1}{3} = \frac{m-1}{1+m} \Rightarrow m = \frac{1}{2}$ (here m is the gradient of line AC)

$\therefore$ Equation of line AC is $y - 1 = \frac{1}{2}(x - 2) \Rightarrow y = \frac{x}{2}$

4. (c) Here $D(1, 1)$ therefore equation of line AD is given by $2x + y - 3 = 0$.

Thus the line perpendicular to AD is $x - 2y + k = 0$ and it passes through B, so $k = 0$.

Hence required equation is $x - 2y = 0$

5. (b) The equation of lines are $y - y_1 = \frac{m \pm \tan \alpha}{1 \mp m \tan \alpha} (x - x_1)$

$$\Rightarrow y - 4 = \frac{1 \pm \tan 45^\circ}{1 \mp \tan 45^\circ} (x - x_1)$$

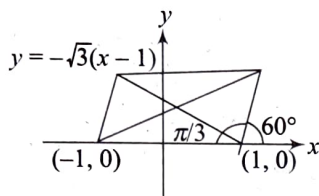
$$\Rightarrow y - 4 = \frac{1 \pm 1}{1 \mp 1} (x - 3)$$

$$\Rightarrow y = 4 \text{ or } x = 3$$

9. (d) Slopes of the line in the new position is b/a , since it is perpendicular to the line $ax + by + c = 0$ and it cuts the x -axis. Hence the required line passes through $(2, 0)$ and its slope is b/a .

The required is $y - 0 = \frac{b}{a}(x - 2)$ or $ay = bx - 2b$

10. (c) $y = 0$ is x -axis
 $y = \sqrt{3}(x - 1)$ is line through point $(1, 0)$ and making an angle 60° with x -axis
 $\sqrt{3}y = (x + 1)$ is line through point $(-1, 0)$ and making an angle 30° with x -axis
 Draw these lines on coordinate axis



From the figure clearly another diagonal passes through the point $(1, 0)$ and having slope $-\sqrt{3}$

$\therefore$ Equation of required diagonal is $y = -\sqrt{3}(x - 1)$

11. (b) Equation of AU is

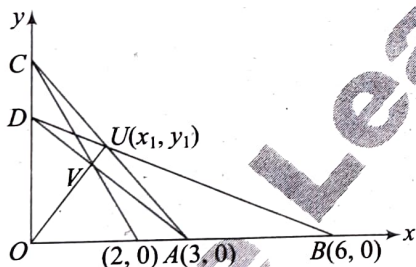
$$y - y_1 = \frac{0 - y_1}{3 - x_1}(x - x_1)$$

So that the coordinates of C are $(0, 3y_1/(3 - x_1))$

Similarly, the coordinates of D are $(0, 6y_1/(6 - x_1))$

Now, equation of AD is $\frac{x}{3} + \frac{y(6 - x_1)}{6y_1} = 1$ (i)

and equation of OU is $y_1x = x_1y$ (ii)



Solving (i) and (ii), we get

$$\frac{x_1y}{3y_1} + \frac{y(6 - x_1)}{6y_1} = 1$$

$$\Rightarrow y(2x_1 + 6 - x_1) = 6y_1$$

$$\Rightarrow y = \frac{6y_1}{6 + x_1} \Rightarrow x = \frac{6x_1}{6 + x_1}$$

Hence, the coordinates of V are $\left(\frac{6x_1}{6 + x_1}, \frac{6y_1}{6 + x_1}\right)$

Therefore, equation of CV is $y - \frac{3y_1}{3 - x_1} = \frac{\frac{6y_1}{6 + x_1} - \frac{3y_1}{3 - x_1}}{\frac{6x_1}{6 + x_1} - 0}(x - 0)$

$$\Rightarrow y = \frac{3y_1}{3 - x_1} - \frac{9x_1y_1}{6x_1(3 - x_1)}x \Rightarrow y = \frac{3y_1}{3 - x_1}\left(1 - \frac{x}{2}\right)$$

which passes through the point $(2, 0)$ for all values of (x_1, y_1)

12. (a) Let x -intercept and y -intercept are a and b respectively.
 Given $2h = a + b$, $k^2 = ab$

Line with intercept a and b is $\frac{x}{a} + \frac{y}{b} = 1$,

It passes through the point $(1, 1)$

$$\therefore \frac{1}{a} + \frac{1}{b} = 1 \therefore a + b = ab$$

$$\therefore 2h = k^2$$

$$\therefore y^2 = 2x$$

13. (c) Equation of the line L is $y = \frac{a}{c}x$

As (h, k) lies on it, hence

$$k = \frac{a}{c}h$$

(i)

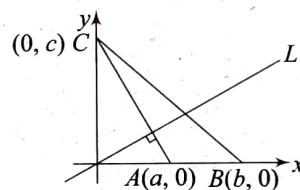
Now equation of BC

$$\frac{x}{b} + \frac{y}{c} = 1$$

(h, k) lies on it

$$\therefore \frac{h}{b} + \frac{k}{c} = 1$$

(ii)

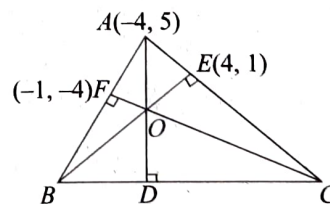


Substituting $c = \frac{ah}{k}$ in (ii) from (i)

$$\frac{h^2}{hb} + \frac{k^2}{ah} = 1$$

$\therefore$ Locus of P is $\frac{x^2}{b} + \frac{y^2}{a} = x$

14. (a)



$$\text{Slope of } AF = \frac{5 - (-4)}{-4 - 1} = \frac{9}{-3} = -3$$

$$\therefore \text{Slope of } CF = \frac{1}{3}$$

$$\therefore \text{Equation of } CF \text{ is } x - 3y - 11 = 0$$

(i)

$$\text{Slope of } AE = \frac{5 - 1}{-4 - 4} = \frac{4}{-8} = -\frac{1}{2}$$

$$\therefore \text{Slope of } BE = 2$$

S.8 Coordinate Geometry

∴ Equation of BE is $2x - y - 7 = 0$

Solving (i) and (ii), we get point $O = (2, -3)$

Equation of AC is $x + 2y - 6 = 0$

Solving (i) and (iii) we get point $C(8, -1)$

Also slope of AO is $-4/3$

∴ Slope of BC is $3/4$

∴ Equation of BC is $3x - 4y - 28 = 0$

15. (c) We have

$$\text{Area } (\Delta OPQ) = \frac{1}{2} \begin{vmatrix} 0 & 0 & 1 \\ a & \frac{2a}{3} & 1 \\ b & \frac{-2b}{3} & 1 \end{vmatrix} = 5 \text{ (Given)}$$

$$\Rightarrow \frac{4ab}{3} = \pm 10$$

So, $4ab = \pm 30$

Also $2h = a + b$

and $2k = \frac{2a - 2b}{3}$ or $a - b = 3k$

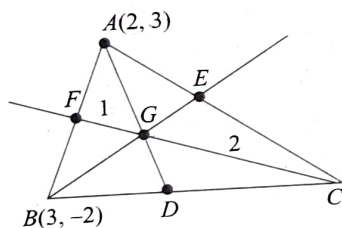
As $4ab = (a + b)^2 - (a - b)^2$

$$\Rightarrow \pm 30 = 4h^2 - 9k^2$$

[Using (i), (ii) and (iii)]

So, required locus can be $4x^2 - 9y^2 = \pm 30$.

16. (b), 17. (a)



Centroid is the point of intersection of $x + y - 1 = 0$; $2y - 1 = 0$

$$\text{or } D\left(\frac{1}{2}, \frac{1}{2}\right)$$

$$\text{Slope of AD is } \frac{3 - \frac{1}{2}}{2 - \frac{1}{2}} = \frac{5}{3}$$

$$\therefore \text{Equation of AD is } y - 3 = \frac{5}{3}(x - 2) \text{ or } 5x - 3y - 1 = 0$$

$$\text{A point on CF: } 2y - 1 = 0 \text{ is } \left(\alpha, \frac{1}{2}\right).$$

Let this be the midpoint of AB.

∴ B is $(2\alpha - 2, -2)$ which lies on median BE: $x + y - 1 = 0$

$$\therefore 2\alpha - 2 - 2 - 1 = 0 \therefore \alpha = \frac{5}{2}$$

$$\therefore B(3, -2)$$

G divides CF in 2 : 1

$$\therefore C = \left(\frac{-7}{2}, \frac{1}{2}\right)$$

$$\therefore \text{Equation of BC is } \frac{y + 2}{\frac{1}{2} + 2} = \frac{x - 3}{\frac{-7}{2} - 3}$$

(ii)

$$\text{or } \frac{y + 2}{5} = \frac{x - 3}{-13}$$

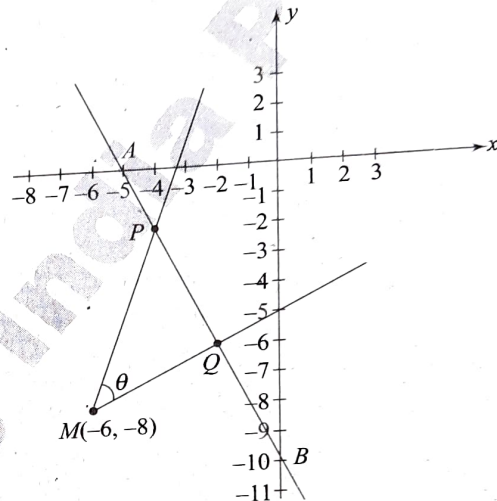
(iii)

$$\text{or } 5x + 13y + 11 = 0$$

DPP 2.2

Single Correct Answer Type

1. (b)



Let line $2x + y + 10 = 0$ intersect axis at $A(-5, 0)$ and $B(0, -10)$

Lines through point $M(-6, -8)$ intersect AB at P and Q.

Slope of AB is $-2 = \tan \alpha$

$$\text{Also } AB = \sqrt{25 + 100} = 5\sqrt{5}$$

$$\text{Given } AP : PQ : QB \equiv 1 : 2 : 2$$

$$\therefore AP = \sqrt{5} \text{ and } AQ = 3\sqrt{5}$$

Using parametric form of straight line AB,

point P is $(-5 - \sqrt{5} \cos \alpha, 0 - \sqrt{5} \sin \alpha)$

$$\text{or } P(-4, -2)$$

Point Q is $(-5 - 3\sqrt{5} \cos \alpha, 0 - 3\sqrt{5} \sin \alpha)$

$$\text{or } Q(-2, -6)$$

∴ Let slopes of PM and QM be m_1 and m_2 , respectively.

$$\therefore m_1 = 3 \text{ and } m_2 = \frac{1}{2}$$

Let θ be the acute angle between PM and QM.

$$\therefore \tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

$$\Rightarrow \tan \theta = 1 \Rightarrow \theta = \frac{\pi}{4}$$

2. (c) Let the line L be $\frac{x}{\cos \theta} = \frac{y}{\sin \theta}$.

$$\text{Then } OP = \frac{5}{2 \cos \theta + 3 \sin \theta}$$

$$\text{and } OQ = \frac{10}{2 \cos \theta + 3 \sin \theta}$$

and let $OR = r$

Then according to condition

$$20 = 6r \cos \theta + 9r \sin \theta$$

$$\therefore \text{locus is } 6x + 9y = 20$$

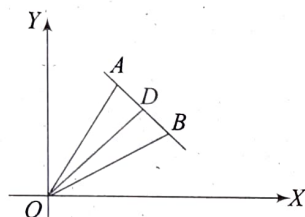
3. (c) As $P(\alpha, (1 + \alpha^2)^{-1})$ lie on $y = \frac{1}{1 + x^2}$ which never intersect the line $y = -2$

$$\therefore \text{On solving } y = \frac{1}{1 + x^2} \text{ with } L_1, \text{ we get } P_1\left(2, \frac{1}{5}\right) \quad (\text{i})$$

$$\text{and with } L_2, \text{ we get } P_2\left(5, \frac{1}{26}\right) \quad (\text{ii})$$

$\therefore$ From (i) and (ii), we get $2 \leq \alpha \leq 5$

4. (b)



$$OA = OB = 9, OD = \frac{15}{\sqrt{25}} = 3$$

$$\therefore AB = 2AD = 2\sqrt{81 - 9} = 2\sqrt{72} = 12\sqrt{2}$$

$$\text{Hence } \Delta = \frac{1}{2}(3 \times 12\sqrt{2}) = 18\sqrt{2} \text{ sq. units.}$$

5. (b) The perpendicular distance of $(1, 3)$ from the line $3x + 4y = 5$ is 2 units while

$$\begin{aligned} \sec^2 \theta + 2 \operatorname{cosec}^2 \theta \\ = 3 + (\tan \theta - \sqrt{2} \cot \theta)^2 + 2\sqrt{2} \\ \geq 3 + 2\sqrt{2}. \end{aligned}$$

Evidently, there will be two such points on the line.

6. (a) Let h be the length of the altitude from A,

$$\text{Distance from centroid to } BC = \frac{h}{3} = \frac{122}{\sqrt{11^2 + 60^2}} = \frac{122}{61} = 2$$

$$\therefore h = 6 \text{ is height of } \Delta ABC$$

$$\therefore \text{Area is } \Delta = \frac{h^2}{\sqrt{3}} = 12\sqrt{3}$$

$$\text{Perimeter, } P = 3 \times \frac{2h}{\sqrt{3}} = 12\sqrt{3}$$

7. (b) Any line through $(0, 0)$ is $y = mx$

By given condition

$$\frac{|\beta - m\alpha|}{\sqrt{1 + m^2}} = d$$

$$(\beta - m\alpha)^2 = d^2(1 + m^2)$$

$$(\beta - (y/x)\alpha)^2 = d^2\left(1 + \frac{y^2}{x^2}\right)$$

$$\Rightarrow \left(\frac{\alpha y - \beta x}{x^2}\right)^2 = d^2(1 + y^2/x^2)$$

$$\Rightarrow (\alpha y - \beta x)^2 = d^2(x^2 + y^2)$$

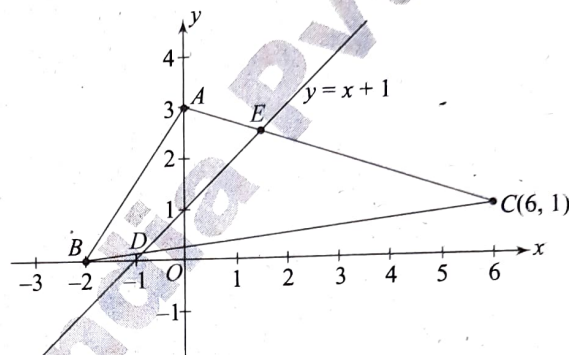
Multiple Correct Answers Type

8. (b, c, d)

We have $A(0, 3)$, $B(-2, 0)$ and $C(6, 1)$

Equation of AC is $x + 3y - 9 = 0$

Equation of BC is $x - 8y + 2 = 0$



$P(\alpha, \alpha + 1)$ lies on the line $y = x + 1$

which cuts BC at $D\left(-\frac{6}{7}, \frac{1}{7}\right)$ and AC at $E\left(\frac{3}{2}, \frac{5}{2}\right)$

$\therefore \alpha$ must lie between $-\frac{6}{7}$ and $\frac{3}{2}$.

9. (a, c) Any point on line through A is

$$(-2 + r \cos \theta, -3 + r \sin \theta)$$

$$\therefore (-2 + AB \cos \theta, -3 + AB \sin \theta) \text{ lies on } x + 3y = 9$$

$$\therefore AB = \frac{20}{(\cos \theta + 3 \sin \theta)}, \text{ similarly } AC = \frac{4}{(\cos \theta + \sin \theta)}$$

$$AB \times AC = 20$$

$$\therefore 4 = \cos^2 \theta + 4 \sin \theta \cos \theta + 3 \sin^2 \theta$$

$$\therefore 4 + 4 \tan^2 \theta = 1 + 4 \tan \theta + 3 \tan^2 \theta$$

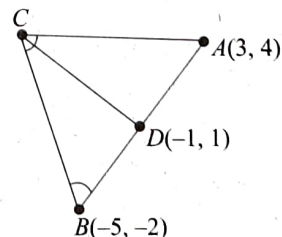
$$\therefore \tan^2 \theta - 4 \tan \theta + 3 = 0$$

$$\therefore \tan \theta = 1 \text{ or } \tan \theta = 3$$

$\therefore$ Required lines are

$$y + 3 = x + 2 \text{ or } y + 3 = 3(x + 2)$$

10. (a, b)



$D(-1, 1)$ is midpoint of AB.

$$CD = BD \cot \frac{C}{2} = 5 \cdot \cot \frac{C}{2}$$

Now, $\tan C = 2$

$$\Rightarrow \frac{2 \tan \frac{C}{2}}{1 - \tan^2 \frac{C}{2}} = 2$$

$$\Rightarrow \tan^2 \frac{C}{2} + \tan \frac{C}{2} - 1 = 0$$

$$\Rightarrow \tan \frac{C}{2} = \frac{-1 + \sqrt{5}}{2}$$

$$\Rightarrow \cot \frac{C}{2} = \frac{\sqrt{5} + 1}{2}$$

$$\Rightarrow CD = 5 \left(\frac{\sqrt{5} + 1}{2} \right) = \frac{5}{2}(\sqrt{5} + 1)$$

Slope of BD is 3/4.

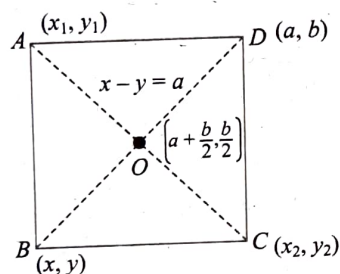
$\therefore$ Slope of CD is $-4/3 = \tan \alpha$

Using parametric form of straight line, coordinates of C are given by

$$\left(-1 \pm \frac{5(\sqrt{5} + 1)}{2} \cos \alpha, 1 \pm \frac{5(\sqrt{5} + 1)}{2} \sin \alpha \right)$$

$$\text{or } \left(-\frac{5 + 3\sqrt{5}}{2}, 2\sqrt{5} + 3 \right) \text{ and } \left(\frac{3\sqrt{5} - 1}{2}, -(2\sqrt{5} + 1) \right)$$

11. (b, d)



Diagonal AC is $x - y = a$

$\therefore$ Slope BD = -1

Also BD passes through (a, b) .

$\therefore$ Equation of BD is $x + y = a + b$

Solving, we get point $O = \left(a + \frac{b}{2}, \frac{b}{2} \right)$

$\therefore$ point B $\equiv (a + b, 0)$

$$m_{AC} = 1 = \tan \theta$$

$$OA = OD = \frac{b}{\sqrt{2}}$$

Using parametric form of straight line points A and C are given by

$$\left(a + \frac{b}{2} \pm \frac{b}{\sqrt{2}} \cos 45^\circ, \frac{b}{2} \pm \frac{b}{\sqrt{2}} \sin 45^\circ \right)$$

or $A(a, 0)$ and $C(a + b, b)$

Single Correct Answer Type

1. (c) Three non-parallel lines are concurrent if $\Delta = 0$

$$\begin{vmatrix} k & 2 & 2 \\ 2 & k & 3 \\ 3 & 3 & k \end{vmatrix} = 0 \Rightarrow k = 2, 3, -5$$

But for $k = 2$, lines are parallel.

2. (b) Lines form triangle. Therefore, $x + 3y + 1 = 0$ is not parallel to $kx + 2y - 2 = 0$

$$\therefore \frac{-k}{2} \neq \frac{-1}{3} \Rightarrow k \neq \frac{2}{3}$$

Also line $2x - y + 3 = 0$ is not parallel to $kx + 2y - 2 = 0$

$$\therefore \frac{-k}{2} \neq 2 \Rightarrow k \neq -4$$

Further lines must not be concurrent

$$\begin{vmatrix} 1 & 3 & 1 \\ k & 2 & -2 \\ 2 & -1 & 3 \end{vmatrix} \neq 0 \Rightarrow k \neq \frac{-6}{5}$$

3. (b) Points which at equal distance from $3x + 4y - 11 = 0$ and $12x + 5y + 2 = 0$ lie on the angle bisector of the lines.

Now given two lines are on opposite sides with respect to origin.

Since required line is near to the origin, we have to find the bisector of the angle containing the origin, which is

$$\frac{3x + 4y - 11}{5} = -\left(\frac{12x + 5y + 2}{13} \right) \text{ or } 99x + 77y - 133 = 0$$

4. (b) Lines making equal angle with given two lines are always parallel to angle bisectors

$$\text{Equation of angle bisectors } \frac{3x - 4y - 1}{5} = \frac{12x + 9y - 1}{15}$$

So bisectors have slope 7, $-\frac{1}{7}$

$$\text{Equation of required lines } y - 1 = 7(x - 1) \text{ and } y - 1 = -\frac{1}{7}(x - 1)$$

which intersect x-axis at $\left(\frac{6}{7}, 0 \right)$ and $(8, 0)$.

5. (c) Line $ax + by + 2 = 0$ always passes through the point $\left(2, \frac{8}{3} \right)$

$$\therefore 6a + 8b + 6 = 0$$

$$\text{or } 3a + 4b + 3 = 0$$

(i)

Now $bx - ay + 4 = 0$ and $3x + 4y + 5 = 0$ meet axis in concyclic points.

So, $m_1 m_2 = 1$

$$\left(\frac{b}{a} \right) \cdot \left(-\frac{3}{4} \right) = 1$$

$$\Rightarrow 4a + 3b = 0 \quad (\text{ii})$$

Solving (i) and (ii), we get $a = 9/7$, $b = -12/7$

$\Rightarrow$ Line $ax + by + 3 = 0$ always passes through the point $(-1, 1)$.

6. (a) $4acx + y(ab + bc + ca - abc) + abc = 0$

Dividing by abc , we get $\frac{4}{b}x + y\frac{3}{b} + 1 - y = 0 \left(\because \frac{2}{b} = \frac{1}{a} + \frac{1}{c} \right)$

$$\therefore \frac{1}{b}(4x + 3y) + 1 - y = 0$$

Lines are concurrent at point of intersection of lines $4x + 3y = 0$ and $1 - y = 0$ or $(-3/4, 1)$

Hence the required line in $y - x = \frac{7}{4}$

7. (c) Equation of the line through AB and AC is

$$(px + qy - 1) + \lambda(qx + py - 1) = 0 \quad (\text{i})$$

$\therefore$ It passes through (p, q) , then

$$(p^2 + q^2 - 1) + \lambda(pq + pq - 1) = 0$$

$$\therefore \lambda = -\frac{(p^2 + q^2 - 1)}{(2pq - 1)}$$

$\therefore$ From equation (i),

$$(px + qy - 1) - \frac{(p^2 + q^2 - 1)}{(2pq - 1)}(qx + py - 1) = 0$$

$$\text{or } (2pq - 1)(px + qy - 1) = (p^2 + q^2 - 1)(qx + py - 1)$$

8. (c) Centroid is $G(0, 0)$

$$\Rightarrow x_1 + x_2 + x_3 = y_1 + y_2 + y_3 = 0$$

$$\text{Given } \sum \frac{(lx_1 + my_1 + n)}{\sqrt{l^2 + m^2}} = 1$$

$$\Rightarrow l^2 + m^2 = 9n^2$$

$$\Rightarrow \frac{l^2 + m^2}{n^2} = 9$$

9. (d) Line passes through the point of intersection of $x - 2y - 2 = 0$ and $2x - by - 6 = 0$

So, its equation is given by $\lambda(x - 2y - 2) + (2x - by - 6) = 0$

As it passes through the origin

$$-2\lambda - 6 = 0 \Rightarrow \lambda = -3$$

$$\therefore \text{Equation of the line is } -x + (6 - b)y = 0$$

$$\text{Its slope is } \frac{1}{6 - b}$$

As its angle with $y = 0$ is less than $\frac{\pi}{4}$

$$\therefore -1 < \frac{1}{6 - b} < 1$$

$$\Rightarrow 6 - b > 1 \text{ or } < -1$$

$$\Rightarrow b < 5 \text{ or } b > 7$$

But $b \neq 4$ (as the lines intersect)

$$\therefore b \in (-\infty, 4) \cup (4, 5) \cup (7, \infty)$$

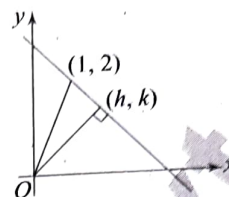
10. (d) Given family of lines is $(4a + 3)x - (a + 1)y - (2a + 1) = 0$

$$\text{or } (3x - y - 1) + a(4x - y - 2) = 0$$

Family of lines passes through the fixed point P which is the intersection of

$$3x - y = 1 \text{ and } 4x - y = 2$$

Solving we get $P(1, 2)$.



Now let (h, k) be the foot of perpendicular on each of the family.

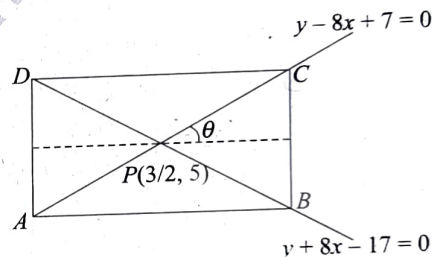
$$\therefore \frac{k}{h} \cdot \frac{k - 2}{h - 1} = -1$$

$$\therefore \text{Locus is } x(x - 1) + y(y - 2) = 0$$

$$\text{or } (2x - 1)^2 + 4(y - 1)^2 = 5$$

Multiple Correct Answers Type

11. (a, c)



The intersection point of the given diagonals is $P \equiv \left(\frac{3}{2}, 5\right)$

Equation of angle bisectors of the diagonals are

$$\frac{y + 8x - 17}{\sqrt{65}} = \pm \frac{y - 8x + 7}{\sqrt{65}}$$

$$\text{or } x = \frac{3}{2} \text{ and } y = 5$$

Let the length of BC be a and that of CD be b .

$$\text{Then } \tan \theta = \frac{a/2}{b/2} = \frac{a}{b} = 8.$$

$$\text{Also } ab = 8$$

$$\Rightarrow a = 8, b = 1.$$

So equations of sides are $y = 1$, $y = 9$, $x = 1$ and $x = 2$.

12. (a, b)

$$6a^2 - 3b^2 - c^2 + 7ab - ac + 4bc = 0$$

$$\therefore (2a + 3b - c)(3a - b + c) = 0$$

$$\Rightarrow -2a - 3b + c = 0 \text{ or } 3a - b + c = 0$$

$$\therefore \text{lines are concurrent at } (-2, -3) \text{ or } (3, -1)$$

13. (b, c, d)

Let image of point $A(\alpha, \beta)$ about $y = 2x$ is $B(a, b)$

$$\Rightarrow \text{mid point of } AB \text{ i.e. } M \equiv \left(\frac{\alpha + a}{2}, \frac{\beta + b}{2}\right) \text{ lie on } y - 2x = 0$$

S.12 Coordinate Geometry

$$\Rightarrow \left(\frac{\beta+b}{2}\right) = 2\left(\frac{\alpha+a}{2}\right) \Rightarrow \beta+b = 2\alpha+2a$$

$$\text{and slope of } AB = -\frac{1}{2}$$

$$\Rightarrow \frac{\beta-b}{\alpha-a} = -\frac{1}{2} \Rightarrow \beta-b = -\frac{1}{2}\alpha + \frac{a}{2}$$

Subtracting (i) and (ii),

$$\Rightarrow 2b = \frac{5}{2}\alpha + \frac{3}{2}a \Rightarrow \alpha = \frac{4b-3a}{5}$$

$$\Rightarrow \beta = \frac{8b-6a}{5} + 2a - b = \frac{3b+4a}{5}$$

$$\Rightarrow (\alpha, \beta) \text{ lies on } xy = 1$$

$$\Rightarrow \left(\frac{4b-3a}{5}\right)\left(\frac{3b+4a}{5}\right) = 1$$

$$\Rightarrow 12b^2 + 7ab - 12a^2 = 25$$

$$\Rightarrow 12x^2 - 7xy - 12y^2 + 25 = 0$$

(i)

(ii)

$\Rightarrow$ Image of $A(4, -1)$ with respect to $x - y - 1 = 0$ is

$$\frac{x-4}{1} = \frac{y+1}{-1} = -\frac{2(4+1-1)}{2} \Rightarrow (0, 3)$$

Image of $A(4, -1)$ with respect to $x - 1 = 0$ is

$$\frac{x-4}{1} = \frac{y+1}{0} = -\frac{2(4-1)}{1} \Rightarrow (-2, -1)$$

Hence, slope of BC is $m = \left(\frac{3+1}{0+2}\right) = 2$

15. (d)

Angle between $x - 1 = 0$ and BC is $\frac{B}{2} \Rightarrow \tan \frac{B}{2} = \frac{1}{2}$

Angle between $x - y - 1 = 0$ and BC is $\frac{C}{2} \Rightarrow \tan \frac{C}{2} = \frac{1}{3}$

$$\Rightarrow \cot \frac{B}{2} \cot \frac{C}{2} = 6$$

Comprehension Type

14. (b)

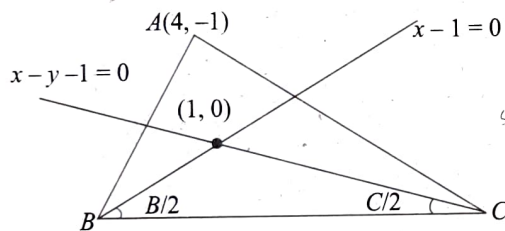


Image of $A(4, -1)$ in bisectors of B and C lies on side BC

CHAPTER 3

DPP 3.1

Single Correct Answer Type

1. (d) We have $9x^2 + 24xy + 16y^2 - 25 = 0$

$$\Rightarrow (3x + 4y)^2 - 25 = 0$$

$$\Rightarrow 3x + 4y + 5 = 0 \quad \text{or} \quad 3x + 4y - 5 = 0$$

These lines are parallel and tangents to the circle

$\therefore$ Distance d between parallel tangents = diameter of the circle

$$\therefore d = \frac{|5 - (-5)|}{\sqrt{3^2 + 4^2}} = 2$$

$$\therefore \text{Radius} = r = 1$$

$$\therefore \text{Area of circle} = \pi \text{ sq. units}$$

2. (b) Given equation is

$$2x^2 - 10xy + 12y^2 + 5x + \lambda y - 3 = 0$$

$$\text{Here } a = 2, h = -5, b = 12, g = \frac{5}{2}, f = \frac{\lambda}{2}, c = -3$$

$$\text{For pair of lines } \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 2 & -5 & 5/2 \\ -5 & 12 & \lambda/2 \\ 5/2 & \lambda/2 & -3 \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 + 25\lambda + 144 = 0$$

$$\Rightarrow \lambda = -9$$

3. (b) Put $y = 0 \Rightarrow 2x^2 - 4x - 6 = 0 \Rightarrow x^2 - 2x - 3 = 0$

$$\Rightarrow (x - 3)(x + 1) = 0$$

$$\therefore \text{Length of intercept} = |3 - (-1)| = 4 \text{ units}$$

4. (c) Homogeneous part of the given equation is $y^2 + 3xy = 0$, which represents straight lines $y = 0$ and $y + 3x = 0$. Now lines perpendicular to these lines are $x = 0$ and $x - 3y = 0$. So combined equation of above lines is

$$x^2 - 3xy = 0$$

5. (b) We have $4y^3 - 8a^2yx^2 - 3ay^2x + 8x^3 = 0$

$$\Rightarrow 4\left(\frac{y}{x}\right)^3 - 8a^2\left(\frac{y}{x}\right)^2 - 8a^2\left(\frac{y}{x}\right) + 8 = 0 \text{ has roots } m_1, m_2, m_3$$

$$\therefore m_1 m_2 m_3 = -2$$

$$\text{Given } m_1 m_2 = -1$$

$$\therefore m_3 = 2$$

$$\therefore 4(2)^3 - 3a(2)^2 - 8a^2(2) + 8 = 0$$

$$\Rightarrow 4a^2 + 3a - 10 = 0$$

$$\therefore \text{Sum of possible values of roots} = \frac{-3}{4}$$

6. (d) We have $f(x, y) \equiv 3x^2 - 4xy + y^2 + 8x - 2y - 3 = 0$
Differentiating w.r.t. x keeping y as constant

$$\frac{\partial f}{\partial x} = 6x - 4y + 8 = 0 \quad \text{(i)}$$

Differentiating w.r.t. y keeping x as constant

$$\frac{\partial f}{\partial y} = -4x + 2y - 2 = 0 \quad \text{(ii)}$$

Solving (i) and (ii), the point of intersection of the pair of lines is $(2, 5)$.

It satisfies the equation $2x - 3y + \lambda = 0$

$$\Rightarrow \lambda = 11$$

7. (b) The joint equation of straight line $y = x - 1$ and $y = -x + 1$ is

$$(x - y - 1)(x + y - 1) = 0$$

$$\Rightarrow x^2 - y^2 - 2x + 1 = 0 \quad \text{(i)}$$

Let equation of line passes through $(2, 0)$ is

$$y = m(x - 2) \quad \text{(ii)}$$

By homogenizing equation (i) with help of line (ii), we get

$$x^2 - y^2 - 2x \left(\frac{mx - y}{2m} \right) + \left(\frac{mx - y}{2m} \right)^2 = 0$$

This is pair of straight lines through origin. Since these lines are perpendicular,

Coefficient of x^2 + coefficient of $y^2 = 0$

$$\Rightarrow m = \pm \frac{1}{\sqrt{3}}$$

8. (a) Any point at distance r from the point $P(1, 2)$ on the line making an angle 45° with the x -axis is $\left(1 + \frac{r}{\sqrt{2}}, 2 + \frac{r}{\sqrt{2}}\right)$

Solving this point with the given pair of lines $x^2 + 4xy + y^2 = 0$,

$$\text{we get } 3r^2 + 9\sqrt{2}r + 13 = 0$$

$$\text{Product of roots } r_1 r_2 = 13/3$$

9. (c) Since $(3, 3)$ is the mid-point of AB .

$\therefore$ We can suppose that the coordinates of A are $(3, 3 + \alpha)$ and coordinates of B are $(3, 3 - \alpha)$.

Slope of $OA = 2 \times (\text{slope of } OB)$

$$\therefore \frac{3 + \alpha}{3} = 2 \frac{3 - \alpha}{3} \Rightarrow \alpha = 1$$

$\therefore$ Combined equation of OA and OB is

$$(2x - 3y)(4x - 3y) = 0$$

$$\text{i.e. } 8x^2 - 18xy + 9y^2 = 0$$

$$\therefore a + 2h + b = 8 - 18 + 9 = -1$$

S.14 Coordinate Geometry

10. (b) Equation of pair of bisectors of angles between lines $ax^2 + 2hxy + by^2 = 0$ is

$$\frac{x^2 - y^2}{xy} = \frac{a - b}{h}$$

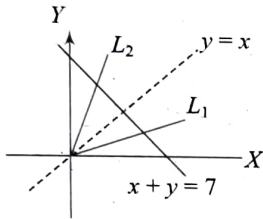
$$\Rightarrow h(x^2 - y^2) = (a - b)xy \quad (i)$$

But $y = mx$ is one of these lines, then it will satisfy it. Substituting $y = mx$ in (i)

$$h(x^2 - m^2x^2) = (a - b)x \cdot mx$$

Dividing by x^2 , we get $h(1 - m^2) = m(a - b)$.

11. (a) $ax^2 + 2hxy + ay^2 = 0$ is symmetrical about $y = x$



Multiple Correct Answers Type

12. (a, b, c, d)

Let lines $x^2 + 2hxy + y^2 = 0$ be given by $y = m_1x$ and $y = m_2x$

$$\therefore m_1 + m_2 = -2h$$

Slope of $y + x = 0$ is -1

$$\therefore \tan \alpha = \left| \frac{m_1 + 1}{1 - m_1} \right| \text{ and } \tan \alpha = \left| \frac{m_2 + 1}{1 - m_2} \right|$$

$$\therefore \tan \alpha = \frac{m_1 + 1}{1 - m_1} \text{ and } \tan \alpha = -\left(\frac{m_2 + 1}{1 - m_2} \right)$$

(for +ve signs, in both gives the same value but $m_1 \neq m_2$).

$$\Rightarrow m_1 = \frac{\tan \alpha - 1}{\tan \alpha + 1}, m_2 = \frac{\tan \alpha + 1}{\tan \alpha - 1}$$

$$\Rightarrow m_1 + m_2 = -2 \sec 2\alpha$$

$$\Rightarrow h = \sec 2\alpha$$

$$\Rightarrow \cos 2\alpha = \frac{1}{h}$$

$$\Rightarrow 2 \cos^2 \alpha - 1 = \frac{1}{h}$$

$$\Rightarrow \cos \alpha = \left(\frac{1+h}{2h} \right)^{\frac{1}{2}} \Rightarrow \cot \alpha = \sqrt{\frac{h+1}{h-1}}$$

13. (b, d)

We must have

$$p \cdot q \cdot 4a + 2 \cdot 2a \cdot 2a \cdot 2\lambda - p \cdot 4a^2 - q \cdot 4a^2 - 4a \cdot 4\lambda^2 = 0$$

$$\Rightarrow 4\lambda^2 - 4a\lambda + \{(p+q)a - pq\} = 0 \quad (\because a \neq 0)$$

$$\text{Since, } \lambda \in R, 16a^2 - 4 \cdot 4 \{(p+q)a - pq\} \geq 0$$

$$\text{or } (a-p)(a-q) \geq 0$$

$$\therefore a \leq p \text{ or } a \geq q$$

14. (a, d)

Since the given equation represents a pair of parallel lines, we have $h^2 = ab \Rightarrow h = \pm 6$

$$\text{Condition for pair of lines } \begin{vmatrix} 9 & h & 3 \\ h & 4 & f \\ 3 & f & -3 \end{vmatrix} = 0$$

$$\Rightarrow 9f^2 - 6hf + 36 = 0$$

$$\text{For } h = 6, f^2 - 4f + 4 = 0 \text{ or } f = 2$$

$$\text{For } h = -6, f^2 + 4f + 4 = 0 \text{ or } f = -2$$

15. (a, c)

$2x^2 + 5xy + 2y^2 + 4x + 5y + a = 0$ represents the pair of lines if

$$\begin{vmatrix} 2 & 5/2 & 2 \\ 5/2 & 2 & 5/2 \\ 2 & 5/2 & a \end{vmatrix} = 0$$

$$\Rightarrow a = 2$$

So pair of lines is $2x^2 + 5xy + 2y^2 + 4x + 5y + 2 = 0$

$$\text{or } x + 2y + 1 = 0, 2x + y + 2 = 0$$

These lines are concurrent with $bx + y + 5 = 0$

$$\text{If } \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ b & 1 & 5 \end{vmatrix} = 0$$

$$\Rightarrow b = 5$$

When lines are concurrent, no circle can be drawn touching all three lines.

16. (c, d)

Lines are perpendicular if $1 + k_1 = 0$

Also for pair of straight line,

$$ABC + 2FGH - AF^2 - BG^2 - CH^2 = 0$$

$$\Rightarrow k_1 = -1 \text{ and } k_2 = \pm a$$

17. (a, b)

$$xy + 4x - 3y - 12 = 0$$

$$\Rightarrow (x-3)(y+4) = 0$$

$$xy - 3x + 4y - 12 = 0$$

$$\Rightarrow (x+4)(y-3) = 0$$

The vertices are

$$A = (-4, -4), B = (-4, 3), C = (3, 3), D = (3, -4)$$

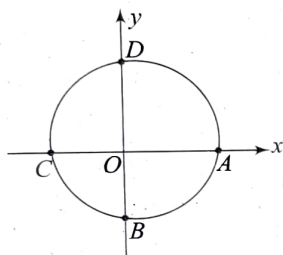
Diagonal AC is $x = y$ and diagonal BD is $x + y + 1 = 0$.

CHAPTER 4

DPP 4.1

Single Correct Answer Type

1. (c) The line $3kx - 2y - 1 = 0$ meets x -axis and y -axis at $A\left(\frac{1}{3k}, 0\right)$ and $B\left(0, -\frac{1}{2}\right)$, respectively. Also, the line $4x - 3y + 2 = 0$ cuts x -axis and y -axis at $C\left(-\frac{1}{2}, 0\right)$ and $D\left(0, \frac{2}{3}\right)$, respectively.



Since the four points are concyclic, we have

$$OB \times OD = OA \times OC$$

$$\Rightarrow (1/2)(2/3) = (1/3k)(1/2)$$

$$\Rightarrow k = 1/2$$

2. (d) $(8, -2)$ lies on the circle $(x - 5)^2 + (y - 2)^2 = 25$ and a chord making a right angle at $(8, -2)$ must be a diameter of the circle. So they all pass through the centre $(5, 2)$.

3. (d) (x, y) lies on the circle with AB as a diameter.

$$\text{Diameter } AB = 2\sqrt{2}$$

$$\therefore \text{Radius} = \sqrt{2}$$

$$\text{Area } (\triangle ABC) = 3$$

$$\Rightarrow \left(\frac{1}{2}\right) (AB) \times (\text{altitude}) = 3$$

$$\Rightarrow \text{Altitude} = \frac{3}{\sqrt{2}} > \text{Radius}$$

Hence, no such C exists.

4. (d) The image of the centre of the given circle i.e., $(-8, 12)$ in the line $4x + 7y + 13 = 0$ is $(-16, -2)$.

$$\text{Also, radius} = \sqrt{64 + 144 - 183} = 5$$

5. (b) $|x - a| + |y - b| = 1$ is square with center at (a, b) which is the center of circle also.

Distance between two parallel sides of square is $\sqrt{2}$.

$$\therefore \text{Radius of the required circle} = \frac{1}{\sqrt{2}}$$

$$\text{Hence, equation of circle is } (x - a)^2 + (y - b)^2 = \frac{1}{2}$$

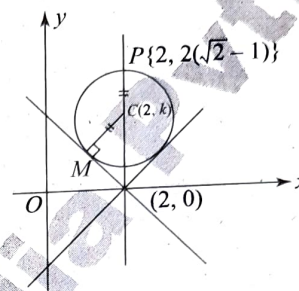
6. (a) The equation of the pair of lines is

$$\Rightarrow (x - 2)^2 - y^2 = 0$$

$$\Rightarrow (x - 2) = \pm y$$

$$\text{i.e. } x - y = 2 \text{ and } x + y = 2$$

The centre of the circle touching the above lines must lie on the angular bisectors of the above lines. Hence, $C \equiv (2, k)$ (see figure).



Thus, we have

$$CM = CP$$

$$\Rightarrow \frac{|2 + k - 2|}{\sqrt{2}} = 2(\sqrt{2} - 1) - k$$

$$\Rightarrow \pm \frac{k}{\sqrt{2}} = 2(\sqrt{2} - 1) - k$$

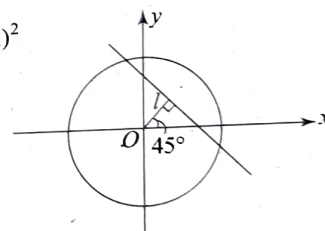
$$\Rightarrow k\left(1 \pm \frac{1}{\sqrt{2}}\right) = 2(\sqrt{2} - 1)$$

$$\begin{aligned} \Rightarrow k &= \frac{2\sqrt{2}(\sqrt{2} - 1)}{\sqrt{2} \pm 1} \\ &= 2\sqrt{2}, 2\sqrt{2}(\sqrt{2} - 1)^2 \\ &= 2\sqrt{2}, 6\sqrt{2} - 8 \end{aligned}$$

7. (a) Centre $\equiv (0, 0)$

$$\text{We must have } \left|\frac{l}{\sqrt{2}}\right| < \sqrt{32}$$

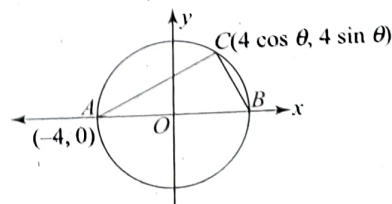
$$\Rightarrow |l| < 8$$



8. (b) Let $P(2 \cos \theta, 2 \sin \theta)$, $Q(-2 \cos \theta, -2 \sin \theta)$

$$\begin{aligned} \therefore \alpha\beta &= \frac{|2 \cos \theta + 2 \sin \theta - 1| \cdot |-2 \cos \theta - 2 \sin \theta - 1|}{2} \\ &= \frac{|4(\cos \theta + \sin \theta)^2 - 1|}{2} \\ &= \frac{|3 + 4 \sin 2\theta|}{2} \\ &\leq \frac{7}{2} \end{aligned}$$

9. (b)



S.16 Coordinate Geometry

$$A = \frac{1}{2} \cdot 8 \cdot 4 \sin \theta = 16 \sin \theta$$

Now, $\sin \theta$ can be equal to $\frac{1}{16}, \frac{2}{16}, \dots, \frac{15}{16}$

i.e. there are 15 points in each quadrant.

10. [d] Let the vertices of the triangle be $(\cos \theta_i, \sin \theta_i)$, $i = 1, 2, 3$.

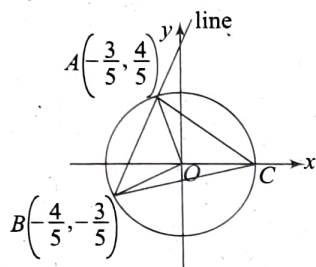
$\therefore$ Circumcenter is $(0, 0)$

Also orthocentre $\equiv ((\cos \theta_1 + \cos \theta_2 + \cos \theta_3), (\sin \theta_1 + \sin \theta_2 + \sin \theta_3))$

$\Rightarrow$ Distance between the orthocentre and the circumcentre

$$\sqrt{(\cos \theta_1 + \cos \theta_2 + \cos \theta_3)^2 + (\sin \theta_1 + \sin \theta_2 + \sin \theta_3)^2} < 3$$

11. (c)



Solving $y = 7x + 5$ and the circle $x^2 + y^2 = 1$, we get

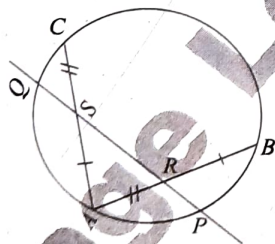
$$A\left(-\frac{3}{5}, \frac{4}{5}\right) \text{ and } B\left(-\frac{4}{5}, -\frac{3}{5}\right)$$

$$m_{OA} = -\frac{4}{3} \text{ and } m_{OB} = \frac{3}{4}$$

Hence, $\angle AOD = 90^\circ \Rightarrow \angle ACB = \frac{\pi}{4}$

12. (b) Perimeter of $\triangle PLM = PL + LC + CM + MP$
 $= PL + LA + BM + MP$
 $= PA + PB.$

13. (a)



Point $A(\sqrt{33} + 3, 0)$ lies on the given circle,

$$x^2 + y^2 - 6x - 8y - 24 = 0.$$

PQ and AB intersect inside the circle.

Let $PR = a$, $RS = b$, $QS = c$

Since $PR \times RQ = AR \times RB$

$$\Rightarrow a(b + c) = 3 \times 7$$

Also, $QS \times SP = 3 \times 7$

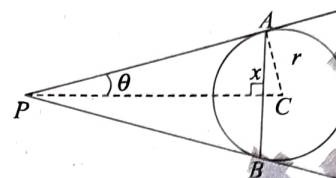
$$\Rightarrow c(a + b) = 3 \times 7$$

$$\Rightarrow a = c$$

$$\therefore PR/QS = 1$$

$$14. (b) \tan \theta = \frac{r}{PA}$$

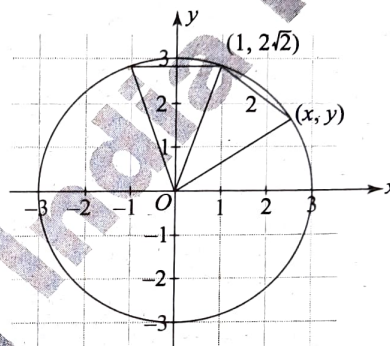
$$\text{Given, } \frac{1}{r^2} + \frac{1}{PA^2} = \frac{1}{16}$$



$$\Rightarrow \frac{\cot^2 \theta + 1}{(PA)^2} = \frac{1}{16}$$

$$\Rightarrow PA \sin \theta = 4 = x \Rightarrow 2x = 8.$$

15. (b)



$$\frac{x_1 - 1}{\cos \theta} = \frac{y_1 - 2\sqrt{2}}{\sin \theta} = 2 \text{ (using parametric form of straight line)}$$

$$\text{Also, } x_1^2 + y_1^2 = 9$$

On solving, we get

$$(1 + 2 \cos \theta)^2 + (2\sqrt{2} + 2 \sin \theta)^2 = 9$$

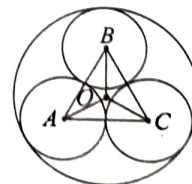
$$\cos \theta = \frac{7}{9} \text{ or } -1$$

Then the required point is $(-1, 2\sqrt{2}), \left(\frac{23}{9}, \frac{10\sqrt{2}}{9}\right).$

DPP 4.2

Single Correct Answer Type

1. (b)



We have $AC = 2a$

and $OA = OC = 1 - a$ (since circles are touching internally)

Since $\triangle ABC$ is equilateral, we have

$$\therefore \angle AOC = 120^\circ$$

Using cosine rule, we have

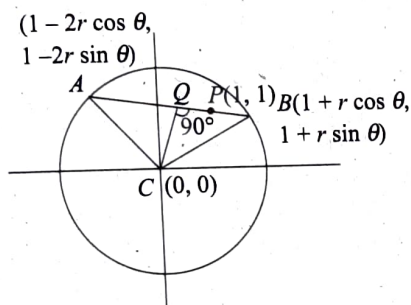
$$\begin{aligned}\cos 120^\circ &= \frac{(1-a)^2 + (1-a)^2 - (2a)^2}{2(1-a)(1-a)} \\ \Rightarrow -\frac{1}{2} &= \frac{-a^2 - 2a + 1}{(1-a)^2} \\ \Rightarrow a^2 - 2a + 1 &= 2a^2 + 4a - 2 \\ \Rightarrow a^2 + 6a - 3 &= 0 \\ \Rightarrow a &= \frac{-6 + \sqrt{48}}{2} = \sqrt{3}(2 - \sqrt{3})\end{aligned}$$

2. (b) Let equation of the circle be $\left(\frac{x^2}{4} + y^2 - 1\right) + \lambda\left(\frac{x^2}{a^2} + y^2 - 1\right) = 0$

$$\begin{aligned}\Rightarrow x^2\left(\frac{1}{4} + \frac{\lambda}{a^2}\right) + y^2(1 + \lambda) &= 1 + \lambda \\ \Rightarrow x^2\left(\frac{a^2 + 4\lambda}{4a^2}\right) + y^2(1 + \lambda) &= 1 + \lambda \\ \Rightarrow x^2\left(\frac{a^2 + 4\lambda}{4a^2(1 + \lambda)}\right) + y^2 &= 1\end{aligned}$$

Clearly, the circle is $x^2 + y^2 = 1$.

3. (d)



Given in $AP : PB = 2 : 1$

Let $AP = 2r$ and $BP = r$

Using parametric equations

$$A = (1 - 2r \cos \theta, 1 - 2r \sin \theta)$$

$$B = (1 + r \cos \theta, 1 + r \sin \theta)$$

A lies on the circle $x^2 + y^2 = 4$

$$\Rightarrow (1 - 2r \cos \theta)^2 + (1 - 2r \sin \theta)^2 = 4$$

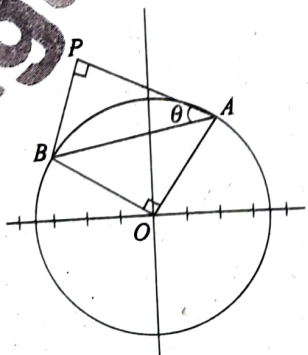
B lies on the circle $x^2 + y^2 = 4$

$$\Rightarrow (1 + r \cos \theta)^2 + (1 + r \sin \theta)^2 = 4$$

Solving (1) and (2), we get $r = 1$.

$$\Rightarrow AB = \sqrt{(3r \cos \theta)^2 + (3r \sin \theta)^2} = 3r = 3$$

4. (d)



$$OA = OB = \text{radius} = \frac{5}{\sqrt{2}} \text{ and } AB = 5$$

$$\Rightarrow \angle AOB = \frac{\pi}{2} \Rightarrow \angle OAB = 45^\circ$$

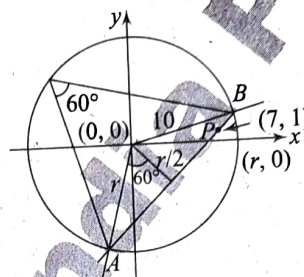
Again $PA = 4$, $PB = 3$ and $AB = 5 \Rightarrow \angle BPA = 90^\circ$

Let $\angle PAB = \theta$.

Then $OP^2 = OA^2 + AP^2 - 2 \cdot OA \cdot AP \cos(45^\circ \pm \theta)$

$$\begin{aligned}&= \frac{25}{2} + 16 - 2 \cdot \frac{5}{\sqrt{2}} \cdot 4 \cdot \frac{1}{\sqrt{2}} (\cos \theta \mp \sin \theta) \\ &= \frac{57}{2} - 20 \left(\frac{4}{5} \mp \frac{3}{5} \right) = \frac{49}{2} \text{ or } \frac{1}{2}\end{aligned}$$

5. (a)



Let the slope of the chord through point $(7, 1)$ be m .

Thus, equation of line is

$$y - 1 = m(x - 7)$$

$$\text{or } mx - y + 1 - 7m = 0$$

Perpendicular distance from $(0, 0) = \frac{r}{2}$

$$\Rightarrow \frac{|7m - 1|}{\sqrt{1 + m^2}} = 5$$

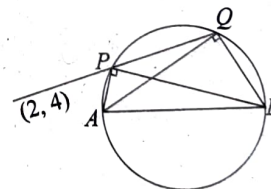
$$\Rightarrow (7m - 1)^2 = 25(1 + m^2)$$

$$\Rightarrow 49m^2 - 14m + 1 = 25 + 25m^2$$

$$\Rightarrow 24m^2 - 14m - 24 = 0$$

$$\Rightarrow m_1 m_2 = -1$$

6. (b) AB subtends right angle at P and Q on variable line.



So, AB is a diameter of circle whose chord is a variable line. Equation of circle is:

$$x(x - 6) + y \times y = 0$$

$$\text{or } x^2 + y^2 - 6x = 0 \quad (i)$$

Equation of line through $(2, 4)$ is:

$$y - 4 = m(x - 2)$$

$$\text{or } y = mx + (4 - 2m) \quad (ii)$$

$$\text{Line (ii) is a chord if } \left| \frac{3m + (4 - 2m)}{\sqrt{1 + m^2}} \right| < 3$$

$$\Rightarrow \left| \frac{4 + m}{\sqrt{1 + m^2}} \right| < 3$$

$$\Rightarrow 16 + 8m + m^2 < 9 + 9m^2$$

S.18 Coordinate Geometry

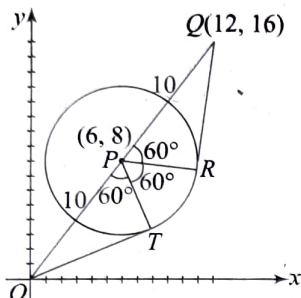
$$\Rightarrow 8m^2 - 8m - 7 > 0$$

$$m \in \left(-\infty, \frac{2-3\sqrt{2}}{4}\right) \cup \left(\frac{2+3\sqrt{2}}{4}, \infty\right)$$

7. (c) Let $O \equiv (0, 0)$, $P \equiv (6, 8)$ and $Q \equiv (12, 16)$.

As shown in the figure, the shortest route consists of tangent OT , minor arc TR and tangent RQ .

Since $OP = 10$, $PT = 5$, and $\angle OTP = 90^\circ$, it follows that $\angle OPT = 60^\circ$ and $OT = 5\sqrt{3}$.



Similarly, $PQ = 10$ and $PR = 5$

$\therefore \angle QPR = 60^\circ$ and $QR = 5\sqrt{3}$

Also points O , P and Q are collinear as $m_{OP} = m_{PQ}$.

Thus, $\angle RPT = 60^\circ$ and arc TR is of length $\frac{5\pi}{3}$.

Hence, the length of the shortest route is $2(5\sqrt{3}) + \frac{5\pi}{3}$.

8. (b) $b^2 + r^2 = (36)^2$ (1)

Also, $CD \cdot CB = CE \cdot CX$

$$\therefore 16 \cdot 36 = (b-r)(b+r)$$

$$\therefore b^2 - r^2 = 16 \times 36$$
 (2)

From (1) and (2),

$$2b^2 = 36(36 + 16) = 36 \times 52$$

$$\Rightarrow b = 6\sqrt{26}$$

9. (a) Chords are of lengths: $l = 1, 2, 3, 4, 5, 6, 7, 8, 7, 6, 5, 4, 3, 2, 1$

$\therefore$ Total number of chords = 15

Length of chord = $2\sqrt{r^2 - d^2}$ (where r is radius and d is distance of chord from center.)

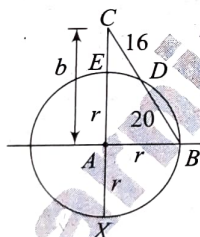
$$\therefore 4(\sum r^2 - \sum d^2) = 2(1^2 + 2^2 + \dots + 7^2) + 8^2$$

$$\Rightarrow 4(\sum r^2 - \sum d^2) = \frac{2 \cdot (7)(8)(15)}{6} + 8^2$$

$$\Rightarrow \sum d^2 = \sum r^2 - \frac{344}{4}$$

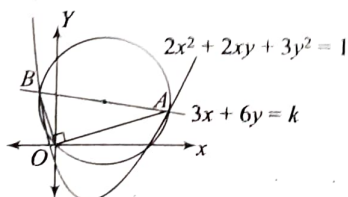
$$\Rightarrow \sum d^2 = 15(16) - 86$$

$$\Rightarrow \sum d^2 = 154$$



Multiple Correct Answers Type

10. (a, d)



$$\text{We have } \frac{3x+6y}{k} = 1 \quad (1)$$

$$2x^2 + 2xy + 3y^2 - 1 = 0 \quad (2)$$

Now homogenising (2) with the help of (1), we get

$$\Rightarrow 2x^2 + 2xy + 3y^2 - \left(\frac{3x+6y}{k}\right)^2 = 0$$

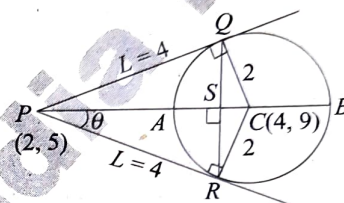
$$\Rightarrow k^2(2x^2 + 3x + 3y^2) - (3x + 6y)^2 = 0$$

Now, coefficient of x^2 + coefficient of $y^2 = 0$

$$\Rightarrow (2k^2 - 9) + (3k^2 - 36) = 0$$

$$\Rightarrow 5k^2 = 45 \Rightarrow k^2 = 9 \Rightarrow k = 3 \text{ or } -3$$

11. (a, b, c, d)



$$\text{Radius} = \sqrt{16 + 81 - 93} = 2$$

$$CP = \sqrt{20}; AP = \sqrt{20} - 2; BP = \sqrt{20} + 2$$

$$\Rightarrow CP = \frac{AP + BP}{2}$$

Thus, CP is the arithmetic mean of AP and BP .

$$\text{Now, let } L = PR = \sqrt{(PC)^2 - r^2} = \sqrt{20 - 4} = 4 = PQ;$$

$$\therefore \tan \theta = \frac{2}{4} = \frac{1}{2}$$

$$\text{Also, } \cos \theta = \frac{PS}{PR}$$

$$\Rightarrow PS = PR \cos \theta = 4 \cdot \left(\frac{2}{\sqrt{5}}\right) = \frac{8}{\sqrt{5}}$$

Harmonic mean between PA and PB

$$= \frac{2(\sqrt{20} - 2)(\sqrt{20} + 2)}{2\sqrt{20}} = \frac{16}{2\sqrt{5}} = \frac{8}{\sqrt{5}} = PS$$

Thus, PS is the harmonic mean of PA and PB .

$$(PS)(CP) = \left(\frac{8}{\sqrt{5}}\right)(\sqrt{20}) = 16 = (PR)^2$$

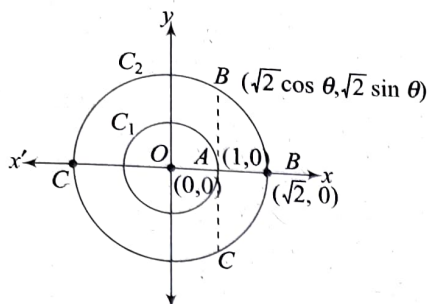
Thus, PR is the geometric mean of PS and CP .

Now, angle between the two tangents.

$$= 2\theta = 2 \tan^{-1}\left(\frac{1}{2}\right) = \tan^{-1}\left(\frac{2 \cdot \frac{1}{2}}{1 - \frac{1}{4}}\right) = \tan^{-1}\left(\frac{4}{3}\right)$$

12. (a, c)

We have maximum $BC = 2\sqrt{2}$ and minimum $BC = 2$.
 $\therefore OA^2 + OB^2 + BC^2 \in [7, 11]$



Let M be the mid-point of AB .

$$\Rightarrow M \equiv \left(\frac{1 + \sqrt{2} \cos \theta}{2}, \frac{\sqrt{2} \sin \theta}{2} \right) \equiv (h, k)$$

$$\therefore \sin \theta = \frac{2k}{\sqrt{2}}, \cos \theta = \frac{2h-1}{\sqrt{2}}$$

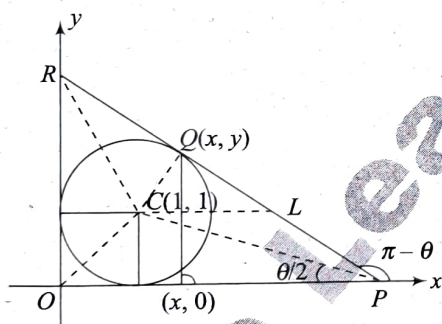
Now, on squaring and adding, we get

$$\Rightarrow 4k^2 + (2h-1)^2 = 2$$

$$\therefore \text{Locus of } M(h, k): \left(x - \frac{1}{2} \right)^2 + y^2 = \left(\frac{1}{\sqrt{2}} \right)^2$$

Comprehension Type

13. (d)



For the circle with center C , using parametric form at straight line, we get

$$x - 1 = \sin \theta$$

$$y - 1 = \cos \theta$$

$$\text{Thus, } Q \equiv (1 + \sin \theta, 1 + \cos \theta)$$

$$14. (c) m_{CQ} = \cot \theta$$

$$m_{PR} = -\tan \theta$$

Equation of PR is

$$y - (1 + \cos \theta) = -\tan \theta (x - (1 + \sin \theta))$$

$$\text{or } y + x \tan \theta = (1 + \cos \theta) + \tan \theta (1 + \sin \theta)$$

$$= \frac{\cos \theta (1 + \cos \theta) + \sin \theta (1 + \sin \theta)}{\cos \theta}$$

$$\text{or } y + x \tan \theta = \frac{1 + \cos \theta + \sin \theta}{\cos \theta}$$

$$\text{or } x \sin \theta + y \cos \theta = (\cos \theta + \sin \theta + 1)$$

$$15. (b) \text{ For } \theta = \pi/4, \text{ equation of } PR \text{ is } x + y = 2 + \sqrt{2}.$$

$$\Rightarrow \text{Area of } \triangle OPR = \frac{1}{2} (2 + \sqrt{2})^2 = (3 + 2\sqrt{2})$$

DPP 4.3

Single Correct Answer Type

$$1. (d) (x - 2) \cos \theta + y \sin \theta = 3 \text{ is tangent to the circle } (x - 2)^2 + y^2 = 3^2.$$

$$2. (c) \text{ Line } 3x - 4y - \lambda = 0 \text{ touches the circle.}$$

$$\Rightarrow \lambda = 15, -35$$

For $\lambda = 15$, point of contact (a, b) is $(5, 0)$.

For $\lambda = -35$, point of contact (a, b) is $(-1, 8)$.

$$3. (b) \text{ Clearly, } (3, 4) \text{ and } (-1, -2) \text{ are end points of diameter.}$$

Thus, Equation of circle is:

$$(x - 3)(x + 1) + (y - 4)(y + 2) = 0$$

$$\text{or } x^2 + y^2 - 2x - 2y - 11 = 0$$

$$4. (c) \text{ The given line is a diameter to the circle.}$$

$$5. (a) x - |y| - \alpha = 0$$

Centre $\equiv (5, 0)$; Radius = 2

Perpendicular distance from centre to the line > Radius

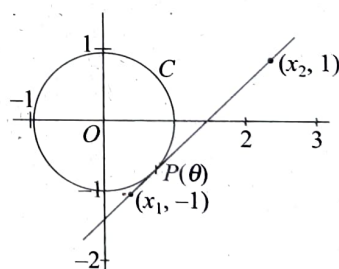
$$\Rightarrow \left| \frac{5 - |0| - \alpha}{\sqrt{1+1}} \right| > 2$$

$$\Rightarrow |5 - \alpha| > 2\sqrt{2}$$

$$\Rightarrow 5 - \alpha > 2\sqrt{2} \quad \text{or} \quad 5 - \alpha < -2\sqrt{2}$$

$$\Rightarrow \alpha < 5 - 2\sqrt{2} \quad \text{or} \quad \alpha > 5 + 2\sqrt{2}$$

6. (a)



$$\text{Equation of circle is } x^2 + y^2 = 1.$$

The equation of tangent to circle at any point $(\cos \theta, \sin \theta)$ is $x \cos \theta + y \sin \theta = 1$.

Since it is passing through points $(x_1, -1)$ and $(x_2, 1)$, we have

$$x_1 \cos \theta - \sin \theta = 1 \quad \text{or} \quad x_1 \cos \theta = 1 + \sin \theta \quad (i)$$

$$\text{and } x_2 \cos \theta = 1 - \sin \theta \quad (ii)$$

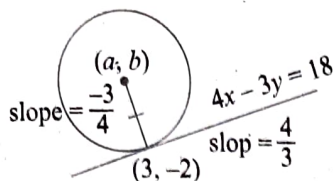
Multiplying the results, we have

$$x_1 x_2 \cos^2 \theta = 1 - \sin^2 \theta$$

$$\Rightarrow x_1 x_2 \cos^2 \theta = \cos^2 \theta$$

$$\Rightarrow x_1 x_2 = 1$$

7. (c)



Let $\frac{x-3}{\cos \theta} = \frac{y+2}{\sin \theta} = 5$; where $\tan \theta = \frac{-3}{4}$

$$\Rightarrow x = 5 \cos \theta + 3$$

$$\text{and } y = 5 \sin \theta - 2$$

$$\text{we have, } \cos \theta = \frac{-4}{5} \text{ and } \sin \theta = \frac{3}{5}$$

$$\therefore x = -1 \text{ and } y = 1$$

$$\Rightarrow \text{centre} \equiv (-1, 1)$$

Hence, required equation of circle is

$$(x+1)^2 + (y-1)^2 = 25$$

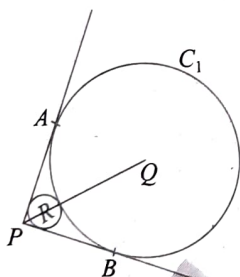
$$\Rightarrow x^2 + y^2 + 2x - 2y - 23 = 0$$

8. (a) The tangents at the origin to C_1 and C_2 are $x + y = 0$, $3x - 4y = 0$, respectively.

Slope of the tangents are -1 and $\frac{3}{4}$, respectively.

Thus if $-1 < m < \frac{3}{4}$, then origin divides AB internally.

9. (a) Let Q be the centre of the given circle and R be the centre of the smallest circle touching the given circle externally. Then



$$AQ = 3 + 2\sqrt{2}$$

$$PQ = 3\sqrt{2} + 4$$

Let radius of smaller circle is r .

$$\Rightarrow 3\sqrt{2} + 4 = QR + RP = 3 + 2\sqrt{2} + r + r\sqrt{2}$$

$$\left(\because \angle APQ = \frac{\pi}{4} \right)$$

$$\Rightarrow (\sqrt{2} + 1) = r(\sqrt{2} + 1) = r = 1$$

10. (d) Any point on the given line can be taken as $\left(\alpha, \frac{3\alpha + 12}{4} \right)$

Then the equation of the chord of contact of tangents from this point to $x^2 + y^2 = 4$ is

$$x\alpha + y \frac{(3\alpha + 12)}{4} - 4 = 0$$

$$\text{or } \alpha(4x + 3y) + (12y - 16) = 0$$

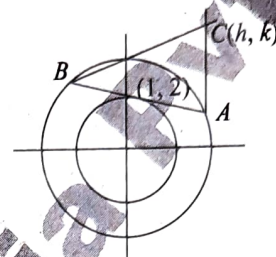
For different values of α , the above chord passes through $P\left(-1, \frac{4}{3}\right)$ which is the point of intersection of $4x + 3y = 0$ and $3y - 4 = 0$.

Now, slope of line joining center of circle $O(0, 0)$ and point

$$P\left(-1, \frac{4}{3}\right) \text{ is } -4/3.$$

Thus, slope of the chord for which P is mid-point is $3/4$.

11. (d)



Tangent at $(1, 2)$ to the circle $x^2 + y^2 = 5$ is:

$$x + 2y - 5 = 0$$

Chord of contact from $C(h, k)$ to $x^2 + y^2 = 9$ is:

$$hx + ky - 9 = 0$$

comparing both equations, we get

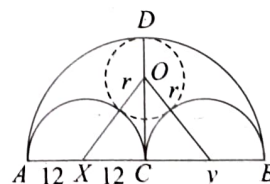
$$\frac{h}{1} = \frac{k}{2} = \frac{9}{5}$$

$$\Rightarrow (h, k) \equiv \left(\frac{9}{5}, \frac{18}{5} \right)$$

DPP 4.4

Single Correct Answer Type

1. (c)



In ΔXCO , we have

$$OX^2 = XC^2 + OC^2$$

$$\Rightarrow (12 + r)^2 = 12^2 + (24 - r)^2$$

$$\Rightarrow r = 8$$

2. (b) If C_3 and C_1 intersect orthogonally then $p - p^2 = 0$

$$\Rightarrow p = 1, 0 \Rightarrow C_2 \text{ represents a circle}$$

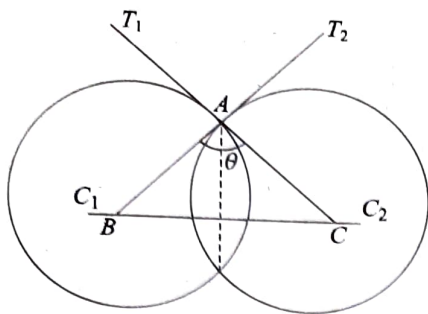
$\Rightarrow$ statement I is false

If C_3 and C_2 intersect orthogonally then $p = -1$

$\Rightarrow$ for this 'p' C_2 and C_3 have equal radii.

$\Rightarrow$ Statement - II is correct

3. (a)



Radii of the circles are same

$$\Rightarrow AB = AC$$

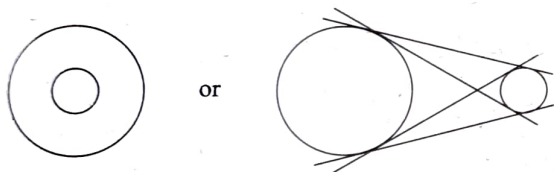
Also, if θ is the angle between the tangents, then

$$\cos \theta = \frac{12 - 4 - 4}{2(2)(2)} = \frac{1}{2}$$

$$\Rightarrow \theta = \frac{\pi}{3}$$

Hence, $\triangle ABC$ is an equilateral triangle.

4. (d) Since C_1 and C_2 in a plane have no points in common, either one circle is completely lying inside other without touching it or two circles are external without intersection or touching as shown in the following figure.

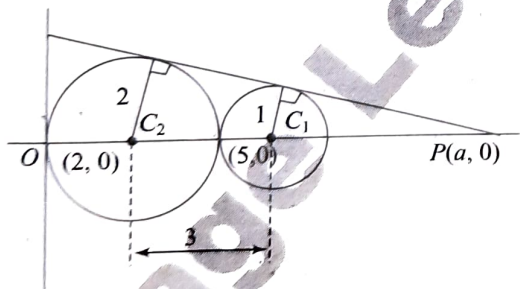


or

5. (b) $\frac{1}{2} = \frac{a-5}{a-2} \Rightarrow a = 8$

Thus, equation of line is $y = m(x - 8)$.

Now, this line touches the circle.



$$\therefore \frac{6m}{\sqrt{1+m^2}} = 2$$

$$\Rightarrow 9m^2 = 1 + m^2$$

$$\therefore m = -\frac{1}{\sqrt{8}}$$

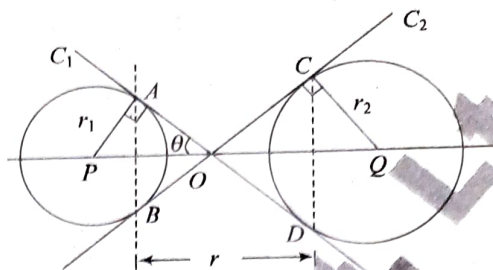
$$\Rightarrow \text{y-intercept} = \sqrt{8} = 2\sqrt{2}$$

6. (b) Both circles cut each other orthogonally.

Thus, C and D will be centres of two circles also CD will be diameter of circumcircle of quadrilateral $ACBD$.

$$\text{Diameter} = \sqrt{3^2 + 4^2} = 5$$

7. (c)

Let $r_1 = 2$ and $r_2 = 4$.

$$\text{Now, } \frac{OP}{OQ} = \frac{r_1}{r_2}$$

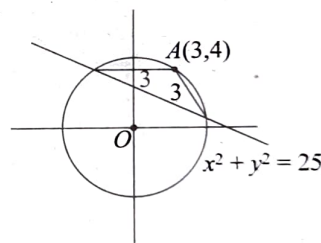
$$OP = \frac{rr_1}{r_1 + r_2}$$

$$OQ = \frac{rr_2}{r_1 + r_2}$$

As triangles OAB and OCD are similar, we have

$$\frac{\text{ar}(\triangle OCD)}{\text{ar}(\triangle OAB)} = \frac{(OC)^2}{(OA)^2} = \left(\frac{\frac{rr_2}{r_1 + r_2} \sin \theta}{\frac{rr_1}{r_1 + r_2} \sin \theta} \right)^2 = \frac{r_2^2}{r_1^2} = 4$$

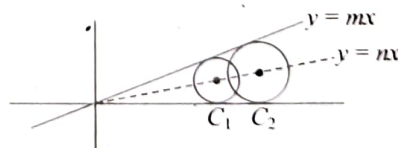
8. (a)

Equation of circle whose centre is at $(3, 4)$ and radius is equal to 3 is $(x - 3)^2 + (y - 4)^2 = 9$.Given circle is $x^2 + y^2 = 25$.

Now, the equation of straight line is the common chord of the two circles.

$$\Rightarrow 6x + 8y = 41$$

9. (a)

Let the centre of C_1 be (a, na) .

$$\Rightarrow (a - 9)^2 + (na - 6)^2 = (na)^2$$

(1)

$$\text{Also, } m = \frac{2n}{1 - n^2}$$

From (1), $a^2 + 117 - 18a - 12na = 0$

$$\Rightarrow a^2 - a(18 + 12n) + 117 = 0$$

$$\Rightarrow a_1 a_2 = 117$$

Let equation of the direct common tangent be $y = m\left(x - \frac{8}{3}\right)$.

By applying condition of tangency, we get

$$\frac{\left|m - \frac{8m}{3}\right|}{\sqrt{1+m^2}} = 1$$

$$\Rightarrow m^2 = \frac{9}{16} \text{ or } m = \pm \frac{3}{4}$$

15. (a, b, c)

Family of circles touching $x + y - 2 = 0$ at $(1, 1)$ is $(x-1)^2 + (y-1)^2 + \lambda(x+y-2) = 0$

$$\Rightarrow x^2 + y^2 + (\lambda-2)x + (\lambda-2)y + 2 - 2\lambda = 0 \quad (1)$$

Given circle is:

$$x^2 + y^2 + 4x + 5y - 6 = 0 \quad (2)$$

Equation of common chord PQ is $S - S' = 0$.

$$\Rightarrow (\lambda-6)x + (\lambda-7)y + 8 - 2\lambda = 0 \quad (3)$$

(a) $PQ \parallel$ line: $x + y - 2 = 0$

$$\Rightarrow \frac{6-\lambda}{\lambda-7} = -1 \Rightarrow 6 = 7; \text{ which is impossible}$$

(b) $PQ \perp$ line: $x + y - 2 = 0$

$$\Rightarrow \frac{6-\lambda}{\lambda-7} = 1 \Rightarrow \lambda = \frac{13}{2}; \text{ which is possible}$$

But when $\lambda = \frac{13}{2}$, we can see that the circles (1) and (2)

are not intersecting each other and their radical axis is perpendicular to the given line $x + y - 2 = 0$.

Eq. (3) can be written as

$$-6x - 7y + 8 + \lambda(x + y - 2) = 0$$

which is in the form $L_1 + \lambda L_2 = 0$

Solving L_1 and L_2 , we get the point of intersection $(6, -4)$.

16. (a, b, d)

$$x^2 + y^2 + \lambda x - 4y + 3 = 0$$

$$\therefore x^2 + y^2 - 4y + 3 + \lambda x = 0$$

Common points on circle are point of intersection of

$$x = 0 \text{ and } y^2 - 4y + 3 = 0$$

So common points are $A(0, 1)$ and $B(0, 3)$.

$\therefore$ Circle has minimum area if AB is diameter

$\therefore$ Equation of circle is

$$x^2 + y^2 - 4y + 3 = 0$$

So radius of director circle is $\sqrt{2}$.

Area of circle $S = 0$ is π sq. units.

Also $|2x| = 1$ never cuts circle.

Comprehension Type

17. (d) Let equation of circle be $(x-r)^2 + (y-r)^2 = r^2$

$\Rightarrow x^2 + y^2 - 2r(x+y) + r^2 = 0$, where r is radius of circle which is passing through (α, β) so

$$\alpha^2 + \beta^2 - 2r(\alpha + \beta) + r^2 = 0$$

$$\Rightarrow r = \frac{2(\alpha + \beta) \pm \sqrt{4(\alpha + \beta)^2 - 4(\alpha^2 + \beta^2)}}{2}$$

$$\Rightarrow r = (\alpha + \beta) \pm \sqrt{2\alpha\beta}$$

18. (a) For orthogonal of two circles $2(r_1 - r_2)^2 = r_1^2 + r_2^2$

$$\Rightarrow (r_1 + r_2)^2 = 6r_1r_2$$

$$\Rightarrow 4(\alpha + \beta)^2 = 6(\alpha^2 + \beta^2)$$

$$\Rightarrow \alpha^2 + \beta^2 = 4\alpha\beta$$

19. (c) $S_1 = x^2 + y^2 - 2r_1(x+y) + r_1^2 = 0$, where $i = 1, 2$

Thus, equation of common chord.

$$2(r_2 - r_1)(x+y) + r_1^2 - r_2^2 = 0$$

$$\Rightarrow 2(x+y) = (r_1 + r_2) = 2(\alpha + \beta)$$

$$\Rightarrow x + y = \alpha + \beta.$$

20. (b), 21. (b)

Equation of any circle through the given points is:

$$(x-a)(x-4a) + (y-5a)(y-a) + \lambda(4x+3y-19a) = 0,$$

for some $\lambda \in \mathbb{R}$.

As it touches the y -axis, putting $x = 0$ and comparing discriminant of resulting quadratic equation in x to 0, we get

$$\therefore \left(-3a + \frac{3\lambda}{2}\right)^2 = 9a^2 - 19\lambda a$$

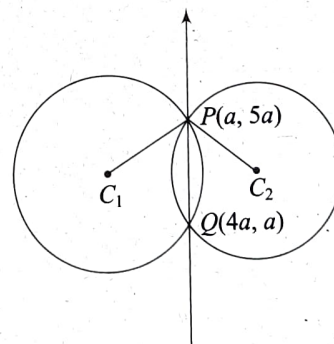
$$\therefore \lambda = 0, \frac{-40a}{9}.$$

$\therefore$ The required circles are

$$x^2 + y^2 - 5ax - 6ay + 9a^2 = 0$$

$$x^2 + y^2 - 5ax - 6ay + 9a^2 - \frac{40a}{9}(4x+3y-19a) = 0$$

Hence centres are $\left(\frac{5a}{2}, 3a\right)$ and $\left(\frac{205a}{18}, \frac{29a}{3}\right)$



The centres of the given circles are:

$$C_1\left(\frac{205a}{18}, \frac{29a}{3}\right) \text{ and } C_2\left(\frac{5a}{2}, 3a\right)$$

Now, the angle of intersection θ of these two circles is the angle between the radius vectors at the common point P to the two circles.

$$\text{i.e., } \angle C_1PC_2 = \theta$$

$$\text{Slope of } C_1P = \frac{\frac{29}{3}a - 5a}{\frac{205}{18}a - a} = \frac{84}{187}$$

and slope of $C_2P = \frac{5a-3a}{a-\frac{5a}{2}} = -\frac{4}{3}$

So, $\tan \theta = \frac{\frac{84}{187} + \frac{4}{3}}{1 - \frac{84}{187} \times \frac{4}{3}} = \frac{252 + 748}{561 - 336} = \frac{40}{9}$

$\Rightarrow \theta = \tan^{-1} \left(\frac{40}{9} \right)$

DPP 4.5

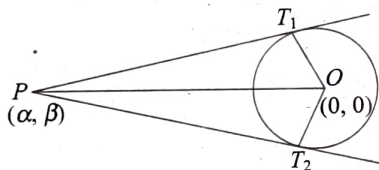
Single Correct Answer Type

1. (d) P, T_2, Q, T_1 are concyclic points with PO as diameter.

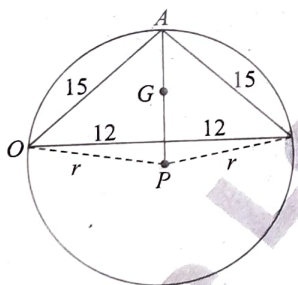
Thus, the circumcentre of ΔPT_1T_2 is $\left(\frac{\alpha}{2}, \frac{\beta}{2} \right)$.

New (α, β) lies on $px + qy + r = 0$

i.e., locus of $\left(\frac{\alpha}{2}, \frac{\beta}{2} \right)$ is $2px + 2qy + r = 0$.



2. (d)



$(r-9)^2 + 12^2 = r^2$
 $\Rightarrow 81 - 18r + 144 = 0$

$\Rightarrow r = \frac{255}{18} = \frac{25}{2}$

Locus is a concentric circle with radius PG .

$PG = (r-9) + \frac{1}{3}(9)$

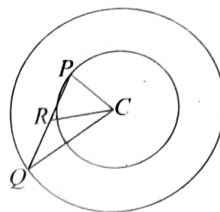
$= r - 6$

$= \frac{25}{2} - 6 = \frac{13}{2}$

Locus is: $(x+1)^2 + (y-1)^2 = \left(\frac{13}{2} \right)^2$

$\Rightarrow x^2 + y^2 + 2x - 2y - \frac{161}{4} = 0$

3. (b) Using Apollonius theorem in ΔPCQ , we get



$CP^2 + CQ^2 = 2(CR^2 + RQ^2)$

$\Rightarrow 16 + 36 = 2 \left(CR^2 + \left(\frac{4}{2} \right)^2 \right)$

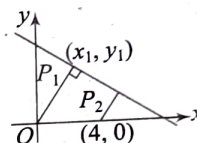
$\Rightarrow 52 = 2(CR^2 + 4)$

$\Rightarrow CR = \sqrt{22}$

4. (a) Let the foot of the perpendicular from origin upon the variable line be (x_1, y_1) .

So, equation of the variable line is:

$xx_1 + yy_1 - (x_1^2 + y_1^2) = 0$



According to the question,

$p_1 p_2 = \left| \frac{x_1^2 + y_1^2}{\sqrt{x_1^2 + y_1^2}} \right| \left| \frac{4x_1 - (x_1^2 + y_1^2)}{\sqrt{x_1^2 + y_1^2}} \right| = 9$

i.e., $|x_1^2 + y_1^2 - 4x_1| = 9$

Locus is: $x^2 + y^2 - 4x - 9 = 0$

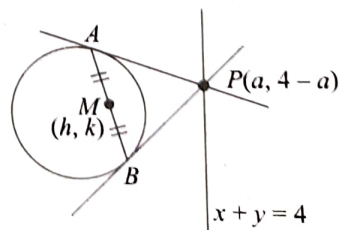
$\therefore r^2 = 4 + 9 = 13$

5. (c) Let the equation of the circle be $x^2 + y^2 = r^2$. The chord which subtends an angle $\frac{\pi}{4}$ at the circumference will subtend a right angle at the centre. So, chord joining $A(r, 0)$ and $B(0, r)$ subtends a right angle at the centre $(0, 0)$. Mid point of AB is

$C \left(\frac{r}{2}, \frac{r}{2} \right)$.

$\therefore OC = \frac{r}{\sqrt{2}}$, which is radius of locus of C .

6. (a) Let $P(a, 4-a)$.



The equation of chord of contact AB is

$xa + y(4-a) = 8$

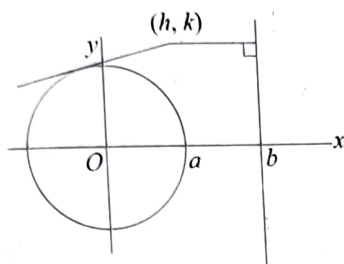
Also, equation of chord AB whose mid-point is $M(h, k)$ is
 $hx + ky = h^2 + k^2$ (ii) (using $T = S_1$)
 Since (i) and (ii) are identical, on comparing the coefficients, we get

$$\frac{a}{h} = \frac{4-a}{k} = \frac{8}{h^2+k^2} = \frac{a+4-a}{h+k} = \frac{4}{h+k}$$

$$\Rightarrow 4(h^2 + k^2) = 8(h + k)$$

$$\text{Hence, locus of } M(h, k) \text{ is } x^2 + y^2 - 2x - 2y = 0.$$

7. (c)



Without the loss of generality, the circle can be taken as $x^2 + y^2 = a^2$ and the given line as $x - b = 0$.

Let (h, k) be the centre of the required circle.

Then length of tangent from (h, k) to the circle and distance of (h, k) from the line should be equal.

$$\text{Hence, } \sqrt{h^2 + k^2 - a^2} = |h - b| \text{ or } k^2 - a^2 = -2bh + b^2$$

$$\text{Thus, the locus is } y^2 - a^2 = -2bx + b^2.$$

$$\text{i.e., } y^2 = -2bx + b^2 + a^2 \text{ which is a parabola.}$$

8. (a) Let the circle be $x^2 + (y - \alpha)^2 = a^2$. Let the point of intersection of tangents at P and Q be (h, k) .

Then equation of PQ , is $hx + (k - \alpha)(y - \alpha) - a^2 = 0$.

As PQ passes through $(a, 0)$, we have

$$ha - \alpha(k - \alpha) - a^2 = 0$$

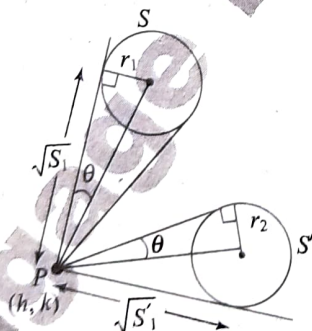
$$\Rightarrow \alpha^2 - k\alpha + a(h - a) = 0$$

Now for real values of α , we have $D \geq 0$.

$$\Rightarrow k^2 - 4a(h - a) \geq 0$$

$$\text{or } y^2 \geq 4a(x - a)$$

9. (b)



Let two circles be $S = 0$ and $S' = 0$ having radii r_1 and r_2 , respectively.

From the figure, we have

$$\frac{r_1}{\sqrt{S_1}} = \frac{r_2}{\sqrt{S'_1}}$$

$$\Rightarrow S'_1 r_1^2 = r_2^2 S_1$$

$$\Rightarrow S_1 - \left(\frac{r_1}{r_2}\right)^2 S'_1 = 0$$

$\therefore$ Locus of $P(h, k)$

$$S - \left(\frac{r_1}{r_2}\right)^2 S' = 0, \text{ which represents the equation of a circle.}$$

10. (a) Let required circle be $x^2 + y^2 + 2gx + 2fy + c = 0$.

Hence, common chord with $x^2 + y^2 - 4 = 0$ is

$$2gx + 2fy + c + 4 = 0$$

This is diameter of circle $x^2 + y^2 = 4$ hence $c = -4$.

Now, again common chord with other circle is:

$$2x(g + 1) + 2y(f - 3) + (c - 1) = 0$$

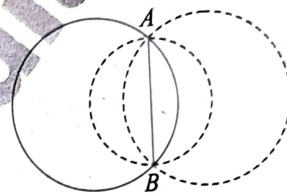
This is diameter of $x^2 + y^2 - 2x + 6y + 1 = 0$.

$$\therefore 2(g + 1) - 6(f - 3) - 5 = 0$$

$$\text{or } 2g - 6f + 15 = 0$$

Locus is $2x - 6y + 15 = 0$, which is straight line.

11. (b)



$$(x^2 + y^2 - 6x - 4y - 12) + \lambda(4x + 3y - 6) = 0$$

This is family of circles passing through points of intersection of circle $x^2 + y^2 - 6x - 4y - 12 = 0$ and line $4x + 3y - 6 = 0$.

Other family will cut this family at A and B .

Hence, locus of centre of circle of other family is this common chord $4x + 3y - 6 = 0$.

Multiple Correct Answers Type

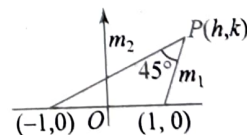
12. (a, d)

$$\text{Let } Q = (3 \cos \theta, 3 \sin \theta) \Rightarrow N = (3 \cos \theta, 0)$$

Point of trisection are $(3 \cos \theta, \sin \theta)$ or $(3 \cos \theta, 2 \sin \theta)$.

$$\text{Locus is } \frac{x^2}{9} + y^2 = 1 \text{ or } \frac{x^2}{9} + \frac{y^2}{4} = 1$$

13. (c, d)



$$\text{Clearly, } m_1 = \frac{k}{h-1} \text{ and } m_2 = \frac{k}{h+1}$$

$$\therefore \tan 45^\circ = \left| \frac{\frac{k}{h-1} - \frac{k}{h+1}}{1 + \frac{k^2}{h^2-1}} \right| = \left| \frac{k(h+1-h-1)}{h^2-1+k^2} \right|$$

$$\Rightarrow h^2 + k^2 - 1 = \pm 2k$$

$$\text{Locus is: } x^2 + y^2 \pm 2y - 1 = 0$$

$$\text{or } x^2 + (y \pm 1)^2 = 2$$

Thus, centre is $(0, 1)$ or $(0, -1)$ and radius is $\sqrt{2}$.

CHAPTER 5

DPP 5.1

Single Correct Answer Type

$$1. (a) \Delta = (1)(1)(2) + 2\left(\frac{3}{2}\right)(0)(-1) - (1)(0)^2 - (1)\left(\frac{3}{2}\right)^2 - 2(-1)^2$$

$$= 2 - \frac{9}{4} - 2 \neq 0$$

$$\text{And } h^2 - ab = 1 - 1 = 0$$

i.e., $h^2 = ab$, so it represents a parabola.

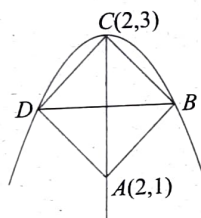
$$2. (c) 3x^2 - 4y + 6x - 3 = 0$$

$$\Rightarrow 3(x+1)^2 = 4y + 6$$

$$\Rightarrow (x+1)^2 = \frac{4}{3}\left(y + \frac{6}{4}\right)$$

$$\Rightarrow \text{L.R.} = \frac{4}{3}$$

3. (b)



AC is parallel to y-axis and its mid-point is (2, 2).

Thus, $B \equiv (3, 2)$ and $D \equiv (1, 2)$.

Equation of parabola,

$$(x-2)^2 = \lambda(y-3)$$

It passes through (3, 2), so

$$1 = -\lambda \Rightarrow \lambda = -1$$

$$\therefore (x-2)^2 = -(y-3)$$

Hence, axis is $x-2=0$, L.R. line is $y-3 = -\frac{1}{4}$

$$\therefore \text{Focus} \equiv \left(2, \frac{11}{4}\right)$$

$$4. (a) \sqrt{x} = \sqrt{a} - \sqrt{y}$$

$$\Rightarrow x = a + y - 2\sqrt{ay}$$

$$\Rightarrow (x-y-a)^2 = 4ay$$

$$\Rightarrow x^2 + (y+a)^2 - 2x(a+y) = 4ay$$

$$\Rightarrow x^2 + y^2 - 2xy + 2ay + a^2 - 2ax = 4ay$$

$$\Rightarrow x^2 + y^2 - 2xy = 2ax + 2ay - a^2$$

$$\Rightarrow (x-y)^2 = 2a\left(x+y-\frac{a}{2}\right)$$

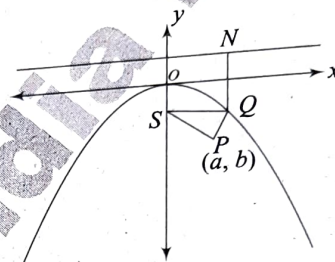
Axis is $x-y=0$

$$\left(\frac{x-y}{\sqrt{2}}\right)^2 = \frac{2a}{2}\left(\frac{x+y-\frac{a}{2}}{\sqrt{2}}\right) \times \sqrt{2}$$

$$\Rightarrow \left(\frac{x-y}{\sqrt{2}}\right)^2 = \sqrt{2}a\left(\frac{x+y-\frac{a}{2}}{\sqrt{2}}\right)$$

$$\therefore \text{Length L.R.} = a\sqrt{2}$$

5. (d)



Let the foot of perpendicular from Q to the directrix be N.
 $SQ + PQ = QN + PQ$ is minimum when P, Q and N are collinear.
 So, minimum value of $SQ + PQ = PN = 1 - b$

6. (b) Equidistant points from focus are symmetric about axis of parabola.

So, tangent at vertex is parallel to the line joining the two points.

$$\therefore \text{Slope} = \frac{3-2}{2-3} = -1$$

7. (b) Equation of circle with AB as diameter is $(x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0$.

Solving it with $y^2 = 4ax$, we get

$$16a^2(y-y_1)(y-y_2) + (y^2-y_1^2)(y^2-y_2^2) = 0$$

$$\Rightarrow (y-y_1)(y-y_2)[16a^2 + (y+y_1)(y+y_2)] = 0$$

$$\Rightarrow (y+y_1)(y+y_2) + 16a^2 = 0$$

$$\Rightarrow y^2 + (y_1+y_2)y + y_1y_2 + 16a^2 = 0$$

The roots of the equation are equal if

$$(y_1+y_2)^2 = 4y_1y_2 + 64a^2 \Rightarrow |y_1 - y_2| = 8a.$$

8. (d) $y^2 = 8x + 16$

$$\Rightarrow y^2 = 8(x+2)$$

Thus, focus is (0, 0).

So, line is $y = \sqrt{3}x$.

Solving this line with parabola,

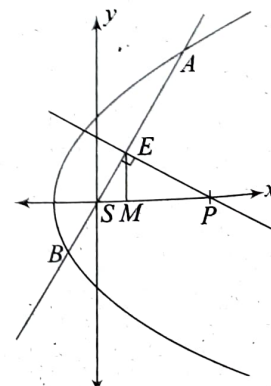
$$3x^2 - 8x - 16 = 0$$

$$\text{i.e., } x_1 + x_2 = \frac{8}{3}$$

$$\Rightarrow \frac{x_1 + x_2}{2} = \frac{4}{3}$$

So, x-coordinate of E is $\frac{4}{3}$.

$$\angle PSE = 60^\circ$$



$$\therefore SP = 2SE$$

$$SE = SM \sec 60^\circ = \frac{8}{3} \therefore SP = 16/3.$$

9. (b) The parabola is $(x-1)^2 = 4\left(y - \frac{1}{2}\right)$ and origin lies outside the parabolic region; $(0, 0)$ makes $x^2 - 2x - 4y + 3$ positive.

$$\therefore (2a, a) \text{ should make } x^2 - 2x - 4y + 3 \text{ negative}$$

$$\text{i.e., } 4a^2 - 8a + 3 < 0$$

$$\Rightarrow (2a-1)(2a-3) < 0$$

$$\text{Thus, } a \text{ belongs to the open interval } \left(\frac{1}{2}, \frac{3}{2}\right).$$

Multiple Correct Answers Type

10. (b, c)

$$S = 0 \Rightarrow x^2 + 2xy + y^2 - 4x - 4y + 4 = 4\lambda(y-x)$$

$$\Rightarrow (x+y-2)^2 = 4\lambda(y-x)$$

If $\lambda = 0$, it represents pair of straight lines otherwise parabola.

11. (b, c)

The points of intersection P, Q, R, S lie on

$$2x^2 + 12x - 8y + 26 = 0 \text{ (by adding equations)}$$

$$\text{i.e., } (x+3)^2 = 4(y-1)$$

So, vertex is $(-3, 1)$ and focus is $(-3, 2)$.

Points of intersection also lie on $y^2 - 20y + 59 = 0$.
(by subtracting equations)

$$\therefore \text{Sum of two of the ordinates} = 20$$

$$\therefore AP + AQ + AR + AS = 40$$

Comprehension Type

12. (a)

13. (a) Any point on circle $x^2 + y^2 = 4$ is $(2 \cos \alpha, 2 \sin \alpha)$.

So, equation of directrix is $x(\cos \alpha) + y(\sin \alpha) - 2 = 0$.

Let focus be (x_1, y_1) .

Then as $A(1, 0), B(-1, 0)$ lie on parabola, we must have

$$(x_1 - 1)^2 + y_1^2 = (\cos \alpha - 2)^2 \text{ (using definition of parabola)}$$

$$\text{and } (x_1 + 1)^2 + y_1^2 = (\cos \alpha + 2)^2$$

$$\Rightarrow x_1 = 2 \cos \alpha, y_1 = \pm \sqrt{3} \sin \alpha$$

Thus, locus of focus is $\frac{x^2}{4} + \frac{y^2}{3} = 1$ and focus is of the form $(2 \cos \alpha, \pm \sqrt{3} \sin \alpha)$.

$\therefore$ Length of semi latus rectum of parabola

= perpendicular distance from focus to directrix $= |2 \pm \sqrt{3}| \sin^2 \alpha$

Hence maximum possible length $= 2 + \sqrt{3}$

DPP 5.2

Single Correct Answer Type

1. (d) $FA = 4$ and $FB = 5$

$$\text{We know that } \frac{1}{a} = \frac{1}{AF} + \frac{1}{FB}$$

$$\Rightarrow a = \frac{20}{9} \Rightarrow 4a = \frac{80}{9}$$

2. (a) Lengths are invariant under change of axes.

Consider $y^2 = 8x$.

Focus is $(2, 0)$. So, focal-chord at distance "2" than vertex is latus rectum which has length "8".

3. (b) $A(0, 2), B = (t_1^2 - 4, t_1), C = (t^2 - 4, t)$

$$\angle CBA = \frac{\pi}{2}$$

$$\Rightarrow \frac{2-t_1}{4-t_1^2} \cdot \frac{t_1-t}{t_1^2-t^2} = -1$$

$$\Rightarrow \frac{1}{2+t_1} \cdot \frac{1}{t+t_1} = -1$$

$$\Rightarrow t_1^2 + (2+t)t_1 + (2t+1) = 0$$

For real t_1 , $(2+t)^2 - 4(2t+1) = t^2 - 4t \geq 0$

$$\Rightarrow t \in (-\infty, 0] \cup [4, \infty)$$

4. (a) Let us take P and Q as t_1 and t_2 respectively.

Equation to PQ is $-2x + y(t_1 + t_2) = 2t_1t_2$

Comparing with $lx + my = 1$, we get

$$t_1 + t_2 = -\frac{2m}{l} \text{ and } t_1t_2 = -\frac{1}{l}$$

$$R \text{ is } \left(\frac{1}{t_1^2}, -\frac{2}{t_1}\right) \text{ and } T \text{ is } \left(\frac{1}{t_2^2}, -\frac{2}{t_2}\right)$$

[$\because$ PSR and QST are focal chords]

$$\begin{aligned} \text{Slope of } RT &= \frac{2\left(\frac{1}{t_2^2} - \frac{1}{t_1^2}\right)}{\frac{1}{t_1^2} - \frac{1}{t_2^2}} = -\frac{2t_1t_2}{t_1 + t_2} \\ &= \frac{-2\left(-\frac{1}{l}\right)}{-\frac{2m}{l}} = -\frac{1}{m} \end{aligned}$$

5. (c) Let $A(2t_1, t_1^2)$ and $B(2t_2, t_2^2)$

$P(-1, 0)$, A and B are collinear.

$$\Rightarrow \frac{t_2^2 - t_1^2}{2(t_2 - t_1)} = \frac{t_2^2}{2t_2 + 1}$$

$$\Rightarrow (t_1 + t_2) = -2t_1t_2 \quad (i)$$

Let centroid be (h, k) .

$$\therefore h = \frac{2(t_1 + t_2)}{3} \quad (ii)$$

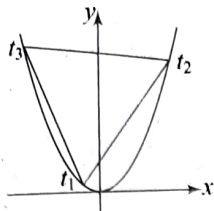
$$\therefore k = \frac{t_1^2 + t_2^2}{3} \quad (iii)$$

From Eqs. (i), (ii), and (iii) eliminating t_1 and t_2 , we get

$$3h^2 + 2h = 4k$$

Hence, locus is $3x^2 + 2x = 4y$.

6. (c)



$$m_1 = 2 = \frac{t_2^2 - t_1^2}{t_2 - t_1}$$

$$\Rightarrow t_1 + t_2 = 2$$

$$\text{Now, } \frac{2 - m_2}{1 + 2m_2} = \sqrt{3}$$

$$\Rightarrow 2 - m_2 = \sqrt{3} + 2\sqrt{3}m_2$$

$$\Rightarrow m_2(2\sqrt{3} + 1) = 2 - \sqrt{3}$$

$$\therefore t_1 + t_3 = m_2 = \frac{2 - \sqrt{3}}{1 + 2\sqrt{3}}$$

$$\text{Further, } \frac{m_3 - 2}{1 + 2m_3} = \sqrt{3}$$

$$\Rightarrow m_3 - 2 = \sqrt{3} + 2\sqrt{3}m_3$$

$$\Rightarrow m_3(1 - 2\sqrt{3}) = 2 + \sqrt{3}$$

$$\Rightarrow m_3 = \frac{2 + \sqrt{3}}{1 - 2\sqrt{3}} = t_2 + t_3$$

Eqs (i), (ii) and (iii), we get

$$\Rightarrow 2(t_1 + t_2 + t_3)$$

$$\begin{aligned} &= 2 + \frac{2 - \sqrt{3}}{1 + 2\sqrt{3}} + \frac{2 + \sqrt{3}}{1 - 2\sqrt{3}} \\ &= 2 + \frac{(2 - \sqrt{3})(2\sqrt{3} - 1) - (2 + \sqrt{3})(2\sqrt{3} + 1)}{11} \\ &= 2 + \frac{(5\sqrt{3} - 8) - (5\sqrt{3} + 8)}{11} \\ &= 2 - \frac{16}{11} = \frac{6}{11} \end{aligned}$$

7. (a) Equation of chord $y = m(x + 6a)$

$$\Rightarrow \frac{y - mx}{6am} = 1$$

Homogenizing with the parabola

$$y^2 - 4ax \left(\frac{y - mx}{6am} \right) = 0$$

$$\Rightarrow 4amx^2 + 6amy^2 - 4axy = 0$$

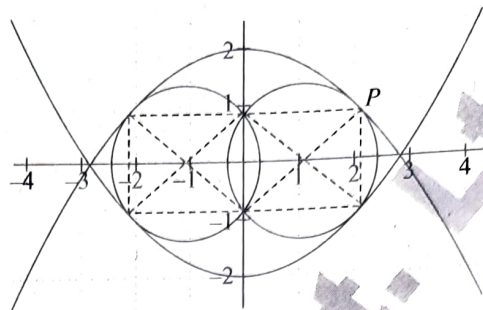
$$\text{Now, } \tan 45^\circ = \left| \frac{2\sqrt{4a^2 - 24a^2m^2}}{4am + 6am} \right|$$

$$\Rightarrow 100a^2m^2 = 4(4a^2 - 24a^2m^2)$$

$$\Rightarrow 49m^2 = 4$$

$$\Rightarrow m = \pm \frac{2}{7}$$

8. (a)



(i)

Let the radius of circle is r .

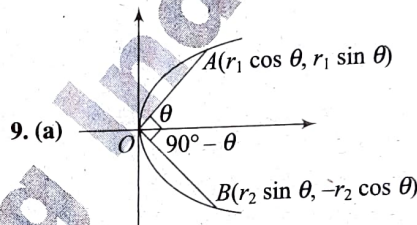
$$\text{From geometry, } P \equiv \left(r\sqrt{2}, \frac{r}{\sqrt{2}} \right)$$

Now, this point lies on the curve $4(2 - y) = x^2$.

$$\Rightarrow 4 \left(2 - \frac{r}{\sqrt{2}} \right) = (r\sqrt{2})^2$$

$$\Rightarrow r = \sqrt{2}$$

(ii)



9. (a)

(iii)

$$r_1^2 \sin^2 \theta = 4ar_1 \cos \theta$$

$$\text{and } r_2^2 \cos^2 \theta = 4ar_2 \sin \theta$$

$$\text{From Eq. (i), we get } r_1^2 \sin^4 \theta = 16a^2 \cos^2 \theta$$

from Eqs. (ii) and (iii), we get

$$\sin^3 \theta = \frac{64a^3}{r_1^2 r_2}$$

$$\text{Similarly, } \cos^3 \theta = \frac{64a^3}{r_1 r_2^2}$$

Eliminating θ , we get

$$\frac{r_1^{4/3} r_2^{4/3}}{r_1^{2/3} + r_2^{2/3}} = 16a^2$$

10. (b) Let $P(-2 + r \cos \theta, r \sin \theta)$.

$$\Rightarrow r^2 \sin^2 \theta - 4r \cos \theta + 8 = 0$$

$$\Rightarrow \frac{1}{AP} + \frac{1}{AQ} = \frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{4}$$

$$\Rightarrow \cos \theta = \frac{1}{2} \Rightarrow \tan \theta = \sqrt{3}$$

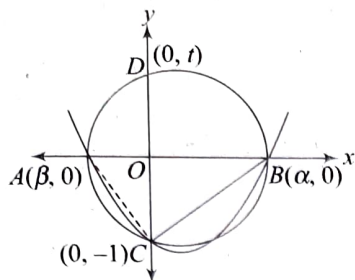
So, equation of line is

$$y - 0 = \sqrt{3} (x + 2)$$

$$\Rightarrow \sqrt{3}x - y + 2\sqrt{3} = 0$$

$$\text{So } \sqrt{a^2 + b^2 + c^2} = 4.$$

11. (b)



$$\text{We have } x = \frac{a \pm \sqrt{a^2 + 4}}{2}$$

$$\therefore \alpha = \frac{a + \sqrt{a^2 + 4}}{2}; \beta = \frac{a - \sqrt{a^2 + 4}}{2}$$

Equation of family of circles through A and B is

$$(x - \alpha)(x - \beta) + y^2 + \lambda y = 0$$

As it passes through C(0, -1), we have

$$\alpha\beta + 1 - \lambda = 0$$

$$\Rightarrow -1 + 1 - \lambda = 0$$

$$\Rightarrow \lambda = 0$$

 $\therefore$ Equation of circle through A, B and C is

$$(x - \alpha)(x - \beta) + y^2 = 0$$

It cuts the y-axis when $x = 0$, so

$$\alpha\beta + y^2 = 0$$

$$y^2 = 1 \text{ (Put } \alpha\beta = -1)$$

$$\Rightarrow y = 1 \text{ or } -1$$

Hence $t = 1$

12. (a) Line PQ is $(t_1 + t_2)y = 2(x + at_1t_2)$

$$\lambda y = -x + b$$

(1) and (2) are same, so

$$\frac{t_1 + t_2}{\lambda} = \frac{2}{-1} = \frac{2at_1t_2}{b}$$

$$\therefore b = -at_1t_2 \text{ and } 2a \leq -at_1t_2 \leq 4a$$

$$\therefore -4 \leq t_1t_2 \leq -2$$

2. (d) The normal at $P(at_1^2, 2at_1)$ is $y + xt_1 = 2at_1 + at_1^3$ with slope say $\tan \alpha = -t_1 = \frac{1}{\sqrt{3}}$.

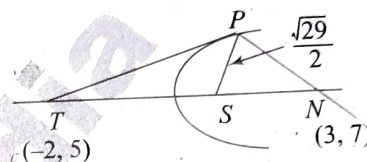
If it meets curve at $Q(at_2^2, 2at_2)$ then $t_2 = -t_1 - \frac{2}{t_1} = -\frac{7}{\sqrt{3}}$.

Then angle θ between parabola (tangent at Q) and normal at P is given by

$$\tan \theta = \left| \frac{-t_1 - \frac{1}{t_2}}{1 - \frac{t_1}{t_2}} \right| = \frac{1}{2\sqrt{3}}$$

$$\Rightarrow \theta = \tan^{-1} \left(\frac{1}{2\sqrt{3}} \right)$$

3. (a)

For parabola $y^2 = 4ax$, tangent and normal at point $P(at^2, 2at)$ meets x-axis at $T(-at^2, 0)$ and $N(2a + at^2, 0)$.

Thus, focus S is the mid-point of TN.

Also, $SP = ST = SN = a + at^2$

$$\therefore SP = TN/2$$

For given data, $TN = \sqrt{29}$

$$\therefore SP = \frac{\sqrt{29}}{2}$$

4. (c) Point of intersection $\equiv (0, -1); \frac{dy}{dx} = \frac{2x}{4}$ and $\frac{-2x}{4}$

$$\therefore m_1 = 0, m_2 = 0 \Rightarrow \theta = 0^\circ$$

5. (d) On solving $y^2 = 4ax$ and $x^2 = 4by$, we get

$$x = 0 \text{ and } x^3 = 64ab^2.$$

$$\left(\frac{dy}{dx} \right)_1 = \left(\frac{2a}{y} \right), \left(\frac{dy}{dx} \right)_2 = \frac{x}{2b}$$

Given curves intersect orthogonally.

$$\text{So, } \frac{2a}{y} \times \frac{x}{2b} = -1$$

$$\Rightarrow ax + by = 0$$

$$\Rightarrow ax + \frac{x^2}{4} = 0$$

(From $x^2 = 4by$)

$$\Rightarrow x = -4a \quad (x \neq 0)$$

$$\Rightarrow -x^3 = 64a^3$$

$$\Rightarrow -64ab^2 = 64a^3$$

$$\Rightarrow a^2 + b^2 = 0$$

which is not possible

6. (b) Let $y = mx + c$ be a common tangent to

$$y = \frac{x^2}{4} - 3x + 10 \quad \text{(i)}$$

$$\text{and } y = 2 - \frac{x^2}{4} \quad \text{(ii)}$$

DPP 5.3

Single Correct Answer Type

1. (a) Solving equation of parabola with x-axis ($y = 0$), we get

$$(a - b)x^2 + (b - c)x + (c - a) = 0,$$

which should have two equal values of x , as x-axis touches the parabola.

$$\therefore (b - c)^2 - 4(a - b)(c - a) = 0$$

$$\Rightarrow (b + c - 2a)^2 = 0$$

$$\Rightarrow (b + c - 2a)^2 = 0$$

$$\Rightarrow -2a + b + c = 0$$

Thus, $ax + by + c = 0$ always passes through $(-2, 1)$.

S.30 Coordinate Geometry

On solving $y = mx + c$ with (i), we get

$$mx + c = \frac{x^2}{4} - 3x + 10$$

$$\text{or } \frac{x^2}{4} - (m+3)x + 10 - c = 0$$

$$D = 0 \text{ gives } (m+3)^2 = 10 - c$$

$$\Rightarrow c = 10 - (m+3)^2$$

Similarly solving line with (ii), we get

$$mx + c = 2 - \frac{x^2}{4}$$

$$\text{or } \frac{x^2}{4} + mx + c - 2 = 0$$

$$D = 0 \text{ gives } m^2 = c - 2$$

$$\Rightarrow c = 2 + m^2$$

From (i) and (iv), we get

$$10 - (m+3)^2 = 2 + m^2$$

$$\Rightarrow 2m^2 + 6m + 1 = 0$$

$$\Rightarrow m_1 + m_2 = -\frac{6}{2} = -3$$

7. (c) $x^2 = 4(y+2)$ is the given parabola.

Any normal is $x = m(y+2) - 2m - m^3$.

If $(10, -1)$ lies on this line then

$$10 = m - 2m - m^3$$

$$\Rightarrow m^3 + m + 10 = 0 \Rightarrow m = -2$$

Slope of normal $= 1/m = -1/2$.

8. (c) Eq. of tangent is $2y = x + 4$.

$$\therefore Q \equiv (-4, 0)$$

Eq. of normal is $y - 4 = -2(x - 4)$ or $y + 2x = 12$.

$$\therefore R \equiv (6, 0)$$

Clearly, QR is diameter of the required circle.

Thus, $(1, 0)$ is required circumcentre.

9. (a) Slope of normal $= \tan 60^\circ = \sqrt{3}$

$\therefore$ Point on normal is $(2(\sqrt{3})^2, -2(2)\sqrt{3})$ or $(6, -4\sqrt{3})$

10. (b) Any point on $x + y = 1$ is $(t, 1-t)$.

Thus, equation of chord with this point as mid point is $y(1-t) - 2a(x+t) = (1-t)^2 - 4at$.

$$-2a(x+t) = (1-t)^2 - 4at$$

It passes through $(a, 2a)$, so $t^2 - t + 2a^2 - 2a + 1 = 0$.

this should have two distinct real roots

$$\therefore a^2 - a < 0 \Rightarrow 0 < a < 1 \Rightarrow 0 < 4a < 4$$

11. (c) Equation of tangent having slope m is $y = mx + \frac{1}{m}$, which passes through the point (h, k) .

$$\therefore m^2h - mk + 1 = 0$$

$$\therefore m_1 + m_2 = \frac{k}{h}; m_1m_2 = \frac{1}{h}$$

$$\text{Given } \theta_1 + \theta_2 = \frac{\pi}{4}$$

$$\Rightarrow \frac{m_1 + m_2}{1 - m_1m_2} = 1$$

$$\Rightarrow \frac{k}{h} = 1 - \frac{1}{h}$$

$$\Rightarrow y = x - 1$$

12. (b) The point of intersection $(at^2, 2at)$ lies on the curve $y^2 + x - y - 2 = 0$.

$$\therefore 4a^2t^2 + at^2 - 2at - 2 = 0$$

$$\Rightarrow (4a^2 + a)t^2 - 2at - 2 = 0$$

$$\text{Sum of roots, } t_1 + t_2 = \frac{2a}{4a^2 + a}$$

$$\text{Product of roots, } t_1t_2 = -\frac{2}{4a^2 + a}$$

Point of intersection of tangents, $(x, y) = (at_1t_2, a(t_1 + t_2))$

$$\therefore x = -\frac{2a}{4a^2 + a}, y = \frac{2a^2}{4a^2 + a}$$

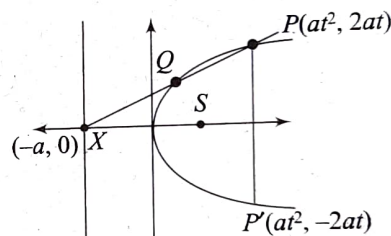
$$\therefore \frac{y}{x} = -a$$

$$\therefore x = -\frac{2\left(-\frac{y}{x}\right)}{4\left(\frac{y}{x}\right)^2 + \left(-\frac{y}{x}\right)}$$

$$\Rightarrow x = \frac{2xy}{4y^2 - xy}$$

$$\Rightarrow 1 = \frac{2}{4y - x} \Rightarrow 4y - x = 2$$

13. (a)



$X: (-a, 0)$

$$PX: y = \frac{2at - 0}{at^2 + a}(x + a)$$

$$\text{or } (1 + t^2)y = 2t(x + a)$$

Solving with parabola $y^2 = 4ax$, we get

$$\frac{4t^2(x+a)^2}{(1+t^2)^2} = 4ax$$

$$\text{or } t^2(x+a)^2 = ax(1+t^2)^2$$

$$\text{or } t^2x^2 - (a + at^4)x + at^2 = 0$$

$$\text{or } xt^2(x - at^2) - a(x - at^2) = 0$$

$$\text{or } x = \frac{a}{t^2}$$

$$\therefore Q: \left(\frac{a}{t^2}, \frac{2a}{t}\right)$$

Thus $P'(at^2, -2at)$ and $Q\left(\frac{a}{t^2}, \frac{2a}{t}\right)$ are extremities of focal chord.

So, tangents at P' and Q intersect on directrix.

14. (b) Let tangent at $P(at^2, 2at)$ make an angle θ with x -axis.

$$\text{Then } \tan \theta = \frac{1}{t}$$

Projection of BC on tangent

$$= BC \sin \theta = \frac{4a}{\sqrt{1+t^2}} \geq 2a\sqrt{2} \quad (\text{as } -1 \leq t \leq 1)$$

15. (b) Any tangent to the parabola is

$$y = mx + \frac{a}{m}$$

Any line through vertex O and perpendicular to tangent is

$$y = -\frac{1}{m}x \quad (\text{ii})$$

OP is perpendicular distance of $O(0, 0)$ from (i).

$$\therefore OP = \frac{a}{\sqrt{1+m^2}} \cdot \frac{1}{m}$$

The line (ii) meets the parabola $y^2 = 4ax$ at Q .

$$\begin{aligned} \therefore \left(-\frac{1}{m}x\right)^2 &= 4ax \\ \Rightarrow x &= 4am^2 \text{ and } y = -4am \end{aligned}$$

So, Q is $(4am^2, -4am)$.

$$\therefore OQ = 4am\sqrt{m^2+1} \quad (\text{using distance formula})$$

$$\begin{aligned} \therefore OP \times OQ &= \left(\frac{a}{\sqrt{1+m^2}} \cdot \frac{1}{m}\right) \cdot 4am\sqrt{1+m^2} \\ &= 4a^2 \end{aligned}$$

Thus $OP, 2a, OQ$ are in G.P.

16. (d) Tangent to parabola $y^2 = 20(x+5)$ having slope m is

$$y = m(x+5) + \frac{5}{m}$$

$$\text{or } m^2(x+5) - my + 5 = 0 \quad (1)$$

Tangent to parabola $y^2 = 60(x+15)$ having slope m' is

$$y = m'(x+15) + \frac{15}{m'} \quad (2)$$

Given $m \times m' = -1$

Thus, (2) reduces to

$$y = -\frac{1}{m}(x+15) - \frac{15}{m} \quad (3)$$

$$\text{or } 15m^2 + my + (x+15) = 0$$

Eliminating m from (1) and (3), we get locus $x+20=0$.

17. (b) Equation of tangent to $y^2 = 36x$ at (x_1, y_1) is

$$yy_1 = 18(x+x_1)$$

It meets axis at $A(-x_1, 0)$ and $B\left(0, \frac{18x_1}{y_1}\right)$.

$$\therefore \text{Centroid of } \triangle OAB, (x, y) = \left(\frac{-x_1}{3}, \frac{6x_1}{y_1}\right)$$

$$\therefore x_1 = -3x \text{ and } y_1 = \frac{6x_1}{y}$$

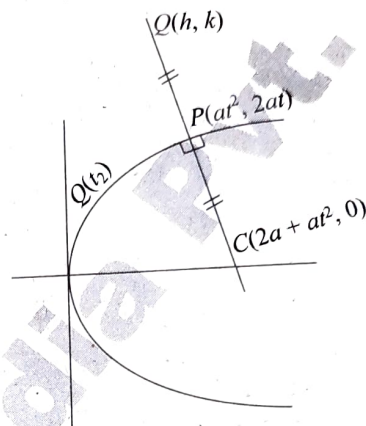
Since, (x_1, y_1) lies on parabola, we have

$$36x_1 = \frac{36x_1^2}{y_1^2}$$

$$\therefore y^2 = x_1 \Rightarrow y^2 = -3x$$

18. (d) For parabola $y^2 = 4ax$, normal at $P(at^2, 2at)$ is

$$y = -tx + 2at + at^3$$



Putting $y = 0$, we get

$$tx = 2at + at^3$$

$$\Rightarrow x = 2a + at^2 = a(2 + t^2)$$

$$\therefore C \equiv (2a + at^2, 0)$$

Since, $PQ = CP$, P is the mid point of QC .

$$\therefore h + 2a + at^2 = 2at^2$$

and $k = 4at$

$$\Rightarrow t = k/4a$$

Putting value of t in (i), we get

$$h + 2a = a\left(\frac{k}{4a}\right)^2$$

$$\Rightarrow h + 2a = \frac{k^2}{16a}$$

$$\Rightarrow y^2 = 16a(x + 2a)$$

which has focus at $(2a, 0)$.

Multiple Correct Answers Type

19. (a, b, c)

If the parabolas have a common normal of slope m ($m \neq 0$) then it is given by

$$y = mx - 2km - km^3$$

$$\text{and } y = m(x-1) - 2m - m^3 = mx - 3m - m^3$$

$$\Rightarrow 2km + km^3 = 3m + m^3$$

$$\Rightarrow m = 0, m^2 = \frac{3-2k}{k-1}$$

If $m^2 < 0$ then the only common normal is the axis.

$$\Rightarrow \frac{3-2k}{k-1} < 0$$

$$\Rightarrow (k-1)(2k-3) > 0$$

$$k > \frac{3}{2} \text{ or } k < 1 \text{ and } k > 0$$

20. (a, b, c)

$$\text{Let } P \equiv (at_1^2, 2at_1)$$

$$\text{Equation of tangent at } P \text{ is } t_1 y = x + at_1^2.$$

$$\text{On } x\text{-axis, } y = 0, \text{ so } Q \equiv (-at_1^2, 0).$$

$$\text{Normal at } P \text{ intersects the parabola at } R(at_2^2, 2at_2).$$

$$\Rightarrow t_2 = -t_1 - \frac{2}{t_1}$$

(i)

$$\text{Slope of } OP, m_1 = \frac{2}{t_1}$$

$$\text{Slope of } OR, m_2 = \frac{2}{t_2}$$

$$\therefore OP \perp OR \Rightarrow t_1 t_2 = -4$$

$$\text{From (i), } t_1 = \pm \sqrt{2}$$

$$\therefore t_2 = \mp 2\sqrt{2}$$

$$PQ = 2at_1 \sqrt{t_1^2 + 1} = 2a\sqrt{6}$$

$$\text{Area of } \Delta PQR = \frac{1}{2} \begin{vmatrix} 2a & 2\sqrt{2}a & 1 \\ -2a & 0 & 1 \\ 8a & -4\sqrt{2}a & 1 \end{vmatrix} = 18\sqrt{2}a^2$$

21. (b, c, d)

The equation to the parabola can be written as:

$$\left(y - \frac{5}{2}\right)^2 = -3\left(x - \frac{25 - 4k}{12}\right)$$

The length of the latus rectum is 3. $y = 2x + 1$ is a tangent. So, the quadratic equation $(2x + 1)^2 - 5(2x + 1) + 3x + k = 0$ or $4x^2 - 3x + k - 4 = 0$ must have equal roots. Now roots are equal if

$$b^2 = 4ac$$

$$\Rightarrow 9 = 16(k - 4) \Rightarrow k = \frac{73}{16}$$

$$y = \frac{5}{2} \text{ is normal at the vertex which has slope } 0.$$

DPP 5.4

Single Correct Answer Type

1. (d) Let ABC be a triangle whose sides are $x = 0$, $y = 0$ and $x + y = 2$.

Since, the parabolas touch the sides, so foci must lie on circumcircle of the triangle ABC whose radius is $\sqrt{2}$.

Now, foci form a triangle of maximum area.

Hence, foci must be the vertices of an equilateral triangle inscribed in the circumcircle.

$$\text{So, area} = \frac{\sqrt{3}}{4} (\sqrt{3}\sqrt{2})^2 = \frac{3\sqrt{3}}{2}$$

2. (c) $t_1 + t_2 + t_3 = 0$

$$\Rightarrow \left(\frac{-2}{3}\right) + 1 + \left(\frac{-k}{15}\right) = 0$$

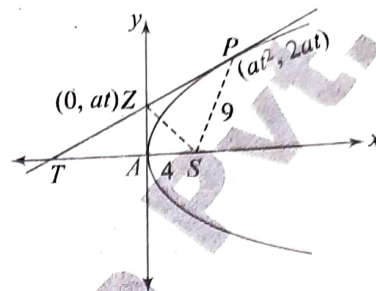
$$\Rightarrow \frac{k}{15} = \frac{1}{3} \Rightarrow k = 5$$

3. (a) Mirror image of focus in the tangent of parabola lies on its directrix.

Here, mirror images of $(2, 3)$ in given lines are $(3, 2)$ and $(-3, -2)$

$$\therefore \text{Equation of directrix is } 2x - 3y = 0.$$

4. (a) Let $P(at^2, 2at)$ be any point on the parabola $y^2 = 4ax$, then the equation of the tangent at P is $yt = x + at^2$. It cuts y -axis at $(0, at)$.



Clearly, SZ is perpendicular to PT .

$$\text{Now, } SZ = \sqrt{a^2 + a^2 t^2} = a\sqrt{1 + t^2}$$

$$SP = a + at^2 \text{ and } AS = a$$

$$\therefore SZ = a^2(1 + t^2) \text{ and } AS \cdot SP = a^2(t^2 + 1)$$

$$\text{Clearly, } SZ^2 = AS \cdot SP$$

So, from the figure, we have

$$\Rightarrow SZ^2 = (4)(9) \Rightarrow SZ = 6$$

5. (A) Let parabola be $y^2 = 4ax$ and co-ordinates of P and Q on this parabola be $P \equiv (at_1^2, 2at_1)$ and $Q \equiv (at_2^2, 2at_2)$. T is the point of intersection of tangents at t_1 and t_2 . Co-ordinates of

$$T \equiv \{at_1 t_2, a(t_1 + t_2)\}$$

$$P' \equiv \{at_1 t_3, a(t_1 + t_3)\}$$

$$Q' \equiv \{at_2 t_3, a(t_2 + t_3)\}$$

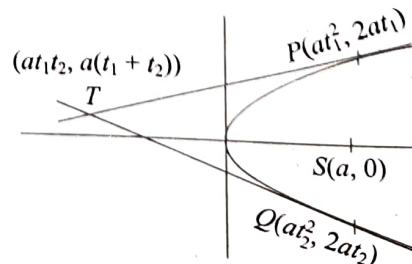
$$\text{Let } TP' : TP = \lambda : 1$$

$$\therefore \lambda = \frac{t_3 - t_2}{t_1 - t_2}$$

$$\Rightarrow \frac{TP'}{TP} = \frac{t_3 - t_2}{t_1 - t_2}$$

$$\text{Similarly, } \frac{TQ'}{TQ} = \frac{t_1 - t_3}{t_1 - t_2}, \frac{TP'}{TP} + \frac{TQ'}{TQ} = 1.$$

6. (b)



If $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$, and $T(at_1 t_2, a(t_1 + t_2))$

Thus, we have $ST^2 = SP \times SQ = 3 \times 12 = 36$

$$\therefore ST = 6$$

7. (b) Let foot of perpendicular from (2, 1) on the directrix be (α, β)

$$\therefore \frac{\alpha-2}{1} = \frac{\beta-1}{1} = \frac{-(2+1+2)}{2}$$

$$\Rightarrow \alpha = -\frac{1}{2}, \beta = -\frac{3}{2}$$

Now, image of point (α, β) in the tangent is focus which is (γ, δ) .

$$\therefore \frac{\gamma + \frac{1}{2}}{2} = \frac{\delta + \frac{3}{2}}{1} = \frac{-2(-1 - \frac{3}{2} - 5)}{5} = 3$$

$$\Rightarrow S = (\gamma, \delta) = \left(\frac{11}{2}, \frac{3}{2}\right)$$

Semi-latus rectum = Distance of focus from directrix

$$= \frac{\left|\frac{11}{2} + \frac{3}{2} + 2\right|}{\sqrt{2}} = \frac{9}{\sqrt{2}}$$

Multiple Correct Answers Type

8. (a, b, c)

Let $A = (2t_1^2, 4t_1)$, $B = (2t_2^2, 4t_2)$ and $C = (2t_3^2, 4t_3)$

Slope of $AB = 2 \Rightarrow t_1 + t_2 = 1$ and $t_1 + t_2 + t_3 = 0$

So, $t_3 = -1$

$$\text{Also, } \frac{2(t_1^2 + t_2^2 + t_3^2)}{3} = \frac{4}{3} \Rightarrow t_1^2 + t_2^2 = 1$$

$$\Rightarrow t_1 = 1, t_2 = 0$$

$A = (2, 4)$, $B = (0, 0)$ and $C = (2, -4)$

Hence $P = (6, 0)$

9. (a, c)

Equation of normal chords at $P(2t_1^2, 4t_1)$ and $R(2t_2^2, 4t_2)$ are

$$y + t_1x - 4t_1 - 2t_1^3 = 0$$

$$\text{and } y + t_2x - 4t_2 - 2t_2^3 = 0.$$

Equation of curve through P, Q, R, S is

$$(y + t_1x - 4t_1 - 2t_1^3)(y + t_2x - 4t_2 - 2t_2^3) + \lambda(y^2 - 8x) = 0$$

P, Q, R, S are concyclic, $t_1 + t_2 = 0$ and $t_1t_2 = 1 + \lambda$.

Thus, Points of intersection of tangents i.e., $(at_1t_2, a(t_1 + t_2))$

lies on X-axis. Slope of $PR = \frac{2}{t_1 + t_2}$

Hence, PR is parallel to Y-axis.

10. (a, b)

Point P' lies on the directrix of $y^2 = 8x$, slopes of PA and PB are 1 and -1 respectively

Equation of $PA: y = x + 2$, Equation of $PB: y = -x - 2$, Equation of $AB: x = 2$

Let $(h, 0)$ be the centre and radius be ' r ' $\Rightarrow \frac{|h+2|}{\sqrt{2}} = \frac{|h-2|}{1} = r$

$$\Rightarrow h^2 - 12h + 4 = 0 \Rightarrow h = 6 \pm 4\sqrt{2}$$

$$\Rightarrow r = |h - 2| = 4(\sqrt{2} - 1), 4(\sqrt{2} + 1)$$

11. (a, b, c, d)

AB is focal chord, tangents at which to the parabola intersect on directrix.

AD is normal chord of the parabola.

12. (a, b, c, d)

$$\text{Mid-point of } PQ = \left(\frac{5}{2}, \frac{1}{2}\right)$$

$$\text{Slope of } OP = \frac{1}{5}$$

$\Rightarrow$ Slope of axis = $\frac{1}{5}$ [$\because$ Line joining the point of intersection of tangents at A and B on parabola and mid-point of chord AB is always parallel to axis of parabola]

$\Rightarrow$ Slope of directrix = -5

So, Equation of directrix is $y = -5x \Rightarrow 5x + y = 0$.

Circle with OP as diameter is $x^2 + y^2 - 3x - 3y = 0$

Circle with OQ as diameter is $x^2 + y^2 - 2x + 2y = 0$.

These circles pass through fixed point focus which is focus

$\left(\frac{30}{13}, \frac{-6}{13}\right)$ which we get by solving above two equations.

Hence, equation of line joining origin to focus is $x + 5y = 0$.

Matching Column Type

13. (a) Normal at A and B intersect at $C(9, 6)$ which lies on parabola. Equation of normal is $y = -tx + 2t + t^3$. Since it passes through C , we have

$$t^3 - 7t - 6 = 0$$

$$\therefore t_1 + t_2 + 3 = 0 \text{ or } t_1 + t_2 = -3;$$

$$t_1t_2t_3 = 6;$$

$$\text{and } t_1t_2 = 2.$$

$$\text{Now, } AB = \sqrt{(t_1^2 - t_2^2)^2 + (2t_1 - 2t_2)^2}$$

$$= \sqrt{(t_1 - t_2)^2[(t_1 + t_2)^2 + 4]}$$

$$= \sqrt{[(t_1 + t_2)^2 - 4t_1t_2][(t_1 + t_2)^2 + 4]}$$

$$= \sqrt{[9 - 8][9 + 4]} = \sqrt{13}$$

$$\begin{aligned} \text{Area of triangle } ABC &= \frac{1}{2} \begin{vmatrix} t_1^2 & 2t_1 & 1 \\ t_2^2 & 2t_2 & 1 \\ 9 & 6 & 1 \end{vmatrix} \\ &= \frac{1}{2} [18(t_1 - t_2) - 6(t_1^2 - t_2^2) + 2t_1t_2(t_1 - t_2)] \\ &= \frac{1}{2} |t_1 - t_2| [18 - 6(t_1 + t_2) + 2t_1t_2] \\ &= 20 \text{ sq. units} \end{aligned}$$

$$\text{Equation of } AB: y - 2t_2 = \frac{2}{t_1 + t_2}(x - t_2^2)$$

$$\text{Or } (t_1 + t_2)y - 2t_1t_2 - 2t_2^2 = 2x - 2t_2^2$$

$$\text{or } 2x + 3y + 4 = 0$$

$$\text{Distance of } AB \text{ from origin is } \frac{4}{\sqrt{13}}.$$

Line AB meets axis at $C(-2, 0)$ and $D(0, -4/3)$

$\therefore$ Area of $\triangle OCD = 4/3$ sq. units

CHAPTER 6

DPP 6.1

Single Correct Answer Type

$$1. (c) \Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 4 & 8 \\ 1 & 8 & 13 \end{vmatrix} = -16 \neq 0$$

$\Rightarrow$ does not represent a pair of straight line.

Again $h^2 = 0$ and $ab = 4 \Rightarrow h^2 < ab$

$\Rightarrow$ ellipse.

$$2. (d) ae = b \Rightarrow e^2 = \frac{b^2}{a^2} \Rightarrow e^2 = 1 - e^2$$

$$\therefore e = \frac{1}{\sqrt{2}}$$

$$\text{Also } SA = a - ae = \sqrt{10} - \sqrt{5}$$

$$\therefore a \left(1 - \frac{1}{\sqrt{2}}\right) = \sqrt{10} \left(1 - \frac{1}{\sqrt{2}}\right) \therefore a^2 = 10 \Rightarrow b^2 = 5$$

$$3. (b) \text{ Foci are } A(-2, 4) \text{ and } B(2, 1)$$

$$\therefore AB = 2ae = 5$$

Center is midpoint of AB which is $C(0, 5/2)$

Distance of center from extremity of minor axis which is $D(1, 23/6)$ is ' b '

$$\therefore b = \frac{5}{3}$$

$$\text{Now, } b^2 = a^2 - a^2e^2$$

$$\Rightarrow a^2 = b^2 + a^2e^2 = \frac{325}{36}$$

$$\Rightarrow e^2 = 1 - \frac{b^2}{a^2} = \frac{3}{\sqrt{13}}$$

$$4. (b) P \text{ lies on the ellipse for which } Q \text{ and } R \text{ are foci}$$

Such that $QR = 2ae = 6$ and $2a =$ perimeter of triangle $-2ae = 16 - 6 = 10$

$$\therefore a = 5 \text{ and } e = 3/5, \text{ also } b^2 = a^2 - a^2e^2 = 25 - 9 = 16$$

$$\therefore b = 4$$

$$\text{Maximum area of triangle} = \frac{1}{2}(2ae)(b) = 12 \text{ sq. units}$$

$$5. (b) 9(x-3)^2 + 9(y-4)^2 = y^2$$

$$\Rightarrow 9(x-3)^2 + 8y^2 - 72y + 144 = 0$$

$$\Rightarrow 9(x-3)^2 + 8(y^2 - 9y) + 144 = 0$$

$$\Rightarrow 9(x-3)^2 + 8 \left[\left(y - \frac{9}{2}\right)^2 - \frac{81}{4} \right] + 144 = 0$$

$$\Rightarrow 9(x-3)^2 + 8 \left(y - \frac{9}{2}\right)^2 = 162 - 144 = 18$$

$$\Rightarrow \frac{9(x-3)^2}{18} + \frac{8 \left(y - \frac{9}{2}\right)^2}{18} = 1$$

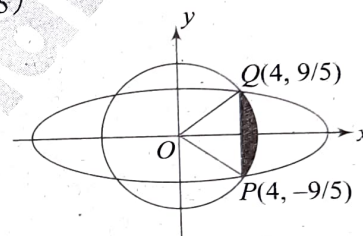
$$\Rightarrow \frac{(x-3)^2}{2} + \frac{\left(y - \frac{9}{2}\right)^2}{9/4} = 1$$

$$\Rightarrow e^2 = 1 - \frac{2 \times 4}{9} = \frac{1}{9} \Rightarrow e = \frac{1}{3}$$

$$6. (a) \text{ As eccentricity of ellipse} = \frac{4}{5}$$

Coordinate of foci = $(4, 0), (-4, 0)$

$\Rightarrow \left(4, -\frac{9}{5}\right)$ is one of the end point of latus-rectum



$\Rightarrow$ Required area is

$$\frac{1}{2\pi} \times \pi \times \left(4^2 + \frac{9^2}{5^2}\right) \times 2 \tan^{-1} \left(\frac{9}{20}\right) - \text{area of } \triangle POQ$$

$$= \frac{481}{25} \tan^{-1} \left(\frac{9}{20}\right) - \frac{1}{2} \times 4 \times \left(\frac{18}{5}\right)$$

$$= \frac{481}{25} \tan^{-1} \left(\frac{9}{20}\right) - \frac{36}{5}$$

$$7. (b) (x-2y+3)^2 + (8x+4y+4)^2 = 20$$

$$\text{or } \frac{\left(\frac{x-2y+3}{\sqrt{5}}\right)^2}{4} + \frac{\left(\frac{2x+y+1}{\sqrt{5}}\right)^2}{\frac{1}{4}} = 1$$

$$\Rightarrow PA + PB = 2a = 4$$

$$8. (b) (4x-8)^2 + 16y^2 = (x+\sqrt{3}y+10)^2$$

$$\Rightarrow (x-2)^2 + y^2 = \left(\frac{1}{2}\right)^2 \frac{(x+\sqrt{3}y+10)^2}{4}$$

Focus is $(h, k) = (2, 0)$, $e = \frac{1}{2}$ and directrix $x + \sqrt{3}y + 10 = 0$

$\frac{a}{e} - ae =$ perpendicular distance from $(2, 0)$ to directrix

$$x + \sqrt{3}y + 10 = 0$$

$$\Rightarrow 2a - \frac{a}{2} = \frac{|2+0+10|}{2}$$

$$\Rightarrow \frac{3a}{2} = 6 \Rightarrow a = 4$$

$$\therefore \text{Distance between directrix} = \frac{2a}{e} = \frac{2.4}{1/2} = 16$$

9. (b) Let equation of secant be

$$\frac{x-2}{\cos \theta} = \frac{y-2}{\sin \theta} = r \text{ (Parametric form)}$$

Solving it with ellipse

$$\frac{(r \cos \theta + 2)^2}{25} + \frac{(r \sin \theta + 2)^2}{16} = 1$$

$$r^2(16 \cos^2 \theta + 25 \sin^2 \theta) + r(64 \cos \theta + 100 \sin \theta) - 236 = 0$$

$$|r_1 r_2| = PA \cdot PB = \left| \frac{-236}{16 \cos^2 \theta + 25 \sin^2 \theta} \right|$$

$$= \left| \frac{236}{16 + 9 \sin^2 \theta} \right|$$

$$\text{Max. } PA \cdot PB = \frac{236}{16} = \frac{59}{4}$$

10. (a) Any point on ellipse $x^2 + 2y^2 = 2$ is

$$x = \sqrt{2} \cos \theta, y = \sin \theta$$

$$\alpha = x^2 + y^2 + \sqrt{2} xy - 1$$

$$= 2 \cos^2 \theta + \sin^2 \theta + \sin 2\theta - 1$$

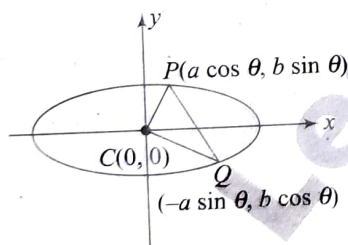
$$= \frac{1 + \cos 2\theta}{2} + \sin 2\theta$$

$$\alpha_{\max} = \frac{\sqrt{5} + 1}{2}$$

11. (c) If P is the point $(a \cos \theta, b \sin \theta)$, the coordinates of Q are

$$\left(a \cos \left(\theta + \frac{\pi}{2} \right), b \sin \left(\theta + \frac{\pi}{2} \right) \right)$$

$$\text{i.e., } (-a \sin \theta, b \cos \theta) \text{ where } a^2 = 28, b^2 = 7$$



$$\therefore \text{The area of } \triangle CPQ = \frac{1}{2} ab \cos^2 \theta + ab \sin^2 \theta$$

$$\left\{ \because \text{area} = \left| \left\{ \frac{1}{2} (x_1 y_2 - x_2 y_1) \right\} \right| \right\}$$

$$= \frac{1}{2} ab$$

$$= \frac{1}{2} (\sqrt{28})(\sqrt{7}) = 7 \text{ sq. units.}$$

12. (b) Since the eccentric angles of P and Q differ by a right angle, we can take P as $(a \cos \theta, b \sin \theta)$ and Q as $(-a \sin \theta, b \cos \theta)$

$$\text{Slope of } CP = \frac{b \sin \theta}{a \cos \theta}$$

$$\text{Slope of } CQ = -\frac{b \cos \theta}{a \sin \theta}$$

If Q is the angle between CP and CQ

$$A = \tan^{-1} \left| \frac{\frac{b \sin \theta}{a \cos \theta} + \frac{b \cos \theta}{a \sin \theta}}{1 - \frac{b^2 \sin \theta \cos \theta}{a^2 \cos \theta \sin \theta}} \right| = \tan^{-1} \left| \frac{2ab}{a^2 - b^2 \sin 2\theta} \right|$$

Q is minimum, if $\sin 2\theta$ is maximum.

$$\text{i.e., if } 2\theta = \frac{\pi}{2} \text{ or } \theta = \frac{\pi}{4}$$

13. (a) According to question $\tan \phi = \frac{b}{a} \tan \theta$

$$\tan(\theta - \phi) = \frac{\tan \theta - \tan \phi}{1 + \tan \theta \tan \phi}$$

$$= \frac{\tan \theta - \frac{b}{a} \tan \theta}{1 + \tan \theta \left(\frac{b}{a} \tan \theta \right)} = \frac{a - b}{a + b \tan^2 \theta}$$

$$\text{This will be maximum if } \tan^2 \theta = \frac{a}{b} \Rightarrow \tan \theta = \sqrt{\frac{a}{b}}$$

14. (a) Ellipse $16x^2 + 25y^2 = 400$

$$\Rightarrow \frac{x^2}{25} + \frac{y^2}{16} = 1$$

Let the points on ellipse be $P(5 \cos \theta_1, 4 \sin \theta_1)$ and $Q(5 \cos \theta_2, 4 \sin \theta_2)$

Circle described on PQ as diameter touches x -axis $(3, 0)$

$$5 \left(\frac{\cos \theta_1 + \cos \theta_2}{2} \right) = 3 \text{ and } 4 \left(\frac{\sin \theta_1 + \sin \theta_2}{2} \right) = r$$

$$\Rightarrow \frac{4}{5} \tan \left(\frac{\theta_1 + \theta_2}{2} \right) = \frac{r}{3} \Rightarrow \tan \left(\frac{\theta_1 + \theta_2}{2} \right) = \frac{5r}{12}$$

$$\Rightarrow \text{Slope of } PQ = -\frac{4}{5} \cot \frac{\theta_1 + \theta_2}{2} = -\frac{48}{25r} = -1$$

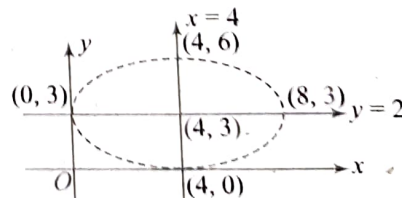
15. (b) Let $(4 \cos \theta + 4, 3 \sin \theta + 3)$ be any point on the ellipse

$$\frac{(x-4)^2}{16} + \frac{(y-3)^2}{9} = 1$$

Image of $(4 \cos \theta + 4, 3 \sin \theta + 3)$ about the line $x - y - 2 = 0$ is (h, k) , then

$$\frac{h - 4 \cos \theta - 4}{1} = \frac{k - 3 \sin \theta - 3}{-1} = -\frac{2(4 \cos \theta - 3 \sin \theta - 1)}{2}$$

$$\Rightarrow h = 3 \sin \theta + 5 \text{ and } k = 4 \cos \theta + 2$$



$$\text{Locus of } (h, k) \text{ is } \left(\frac{x-5}{3} \right)^2 + \left(\frac{y-2}{4} \right)^2 = 1$$

$$\Rightarrow 16x^2 + 9y^2 - 160x - 36y + 292 = 0$$

$$\Rightarrow k_1 + k_2 = 25$$

16. (d) If PSS' forms a triangle, then ellipse

$$\therefore PS + PS' > SS'$$

$$\Rightarrow K \in (-2, 2)$$

If PSS' is collinear, the pair of coincident lines $\Rightarrow K = \pm 2$

If PSS' neither forms triangle nor collinear then empty set

$$\Rightarrow PS + PS' < SS'$$

$$\Rightarrow K \in (-\infty, -2) \cup (2, \infty)$$

17. (d) Let ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ Mid-point of PS is (h, k)

$$\therefore h = \frac{a \cos \theta + ae}{2} \Rightarrow \cos \theta = \frac{2h - ae}{a}$$

$$\text{and } k = \frac{b \sin \theta}{2}$$

Eliminating θ , we get locus as

$$\frac{(2h - ae)^2}{a^2} + \frac{4k^2}{b^2} = 1$$

$$\Rightarrow \frac{\left(x - \frac{ae}{2}\right)^2}{\frac{a^2}{4}} + \frac{y^2}{\frac{b^2}{4}} = 1, \text{ which is ellipse having enclosed area}$$

$$\text{Area} \Rightarrow \pi \frac{a}{2} \cdot \frac{b}{2} = \frac{\pi ab}{4} \therefore \text{ratio} = \frac{1}{4}$$

18. (b) $\left(7 - \frac{5}{4}\alpha, \alpha\right)$ lies inside the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$

$$\Rightarrow \frac{\left(7 - \frac{5}{4}\alpha\right)^2}{25} + \frac{\alpha^2}{16} - 1 < 0$$

$$\Rightarrow \frac{(28 - 5\alpha)^2}{400} + \frac{\alpha^2}{16} - 1 < 0$$

$$\Rightarrow (28 - 5\alpha)^2 + 25\alpha^2 - 400 < 0$$

$$\Rightarrow 50\alpha^2 - 280\alpha - 384 < 0$$

$$\Rightarrow 25\alpha^2 - 140\alpha - 192 < 0$$

$$\Rightarrow (5\alpha - 12)(5\alpha - 16) < 0$$

$$\Rightarrow \frac{12}{5} < \alpha < \frac{16}{5}$$

19. (a) $4 \tan \frac{B}{2} \tan \frac{C}{2} = 4 \sqrt{\frac{(s-a)(s-c)}{s(s-b)}} \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = 1$

$$\Rightarrow 4 \frac{(s-a)}{s} = 1$$

$$\Rightarrow s = \frac{4a}{3} = 4 \times \frac{6}{3} = 8$$

$$\text{Now } 2s = a + b + c = 16$$

$$\Rightarrow b + c = 10$$

Hence locus is an ellipse having center $\equiv (5, 0)$

$$2ae = 6$$

$$\text{And } 2a = 10$$

$$b^2 = a^2 - a^2 e^2 = 25 - 9 = 16$$

$$\therefore \text{Equation of ellipse is } \frac{(x-5)^2}{25} + \frac{y^2}{16} = 1.$$

20. (b) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$P(a \cos 2\alpha, b \sin 2\alpha), Q(a \cos 2\beta, b \sin 2\beta)$$

$$R(a \cos 2\gamma, b \sin 2\gamma)$$

Equation of chord PQ is

$$\frac{x}{a} \cos(\alpha + \beta) + \frac{y}{b} \sin(\alpha + \beta) = \cos(\alpha - \beta)$$

PQ passes through the focus $(ae, 0)$

$$\therefore e = \frac{\cos(\alpha - \beta)}{\cos(\alpha + \beta)}$$

PR passes through the focus $(-ae, 0)$ the

$$\therefore -e = \frac{\cos(\alpha - \gamma)}{\cos(\alpha + \gamma)}$$

$$\therefore \frac{\cos(\alpha - \beta)}{\cos(\alpha + \beta)} = -\frac{\cos(\alpha - \gamma)}{\cos(\alpha + \gamma)}$$

Apply componendo and dividendo, we get

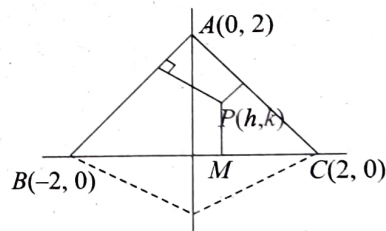
$$\frac{\cos(\alpha + \beta) + \cos(\alpha - \beta)}{\cos(\alpha + \beta) - \cos(\alpha - \beta)} = \frac{\cos(\alpha + \gamma) - \cos(\alpha - \gamma)}{\cos(\alpha + \gamma) + \cos(\alpha - \gamma)}$$

$$\frac{2 \cos \alpha \cos \beta}{2 \sin \alpha \sin \beta} = \frac{2 \sin \alpha \sin \gamma}{2 \cos \alpha \cos \gamma}$$

$$\tan \beta \tan \gamma = \cot^2 \alpha$$

Multiple Correct Answers Type

21. (a, c)



$$PM = k$$

$$\text{Equation of } AB \equiv -x + y = 2$$

$$\text{Equation of } AC \equiv x + y = 2$$

According to question

$$\left(\frac{2-h-k}{\sqrt{2}}\right)\left(\frac{2+h+k}{\sqrt{2}}\right) = 2k^2$$

$$\Rightarrow h^2 + 3k^2 + 4k = 4$$

$$\Rightarrow h^2 + 3\left(k^2 + \frac{4}{3}k + \frac{4}{9}\right) = 4 + \frac{4}{3}$$

$$\Rightarrow h^2 + 3\left(k + \frac{2}{3}\right)^2 = \frac{16}{3} \Rightarrow \frac{h^2}{16/3} + \frac{\left(k + \frac{2}{3}\right)^2}{16/9} = 1$$

$$\Rightarrow \text{Ellipse with } e = \sqrt{\frac{2}{3}} \text{ and } D \equiv \left(0, -\frac{2}{3}\right).$$

22. (a, b)

Let (h, k) be the extremity of L.R.

$$\therefore h = \pm ae; k = \pm \frac{b^2}{a}$$

$$\therefore k = \pm a(1 - e^2) = \pm a \left(1 - \frac{h^2}{a^2}\right) = \pm \left(a - \frac{h^2}{a}\right)$$

On taking +ve sign, we get

$$k = a - \frac{h^2}{a}$$

$$\Rightarrow \frac{h^2}{a} = a - k \Rightarrow a - k \Rightarrow h^2 = a(a - k).$$

On taking -ve sign, we get

$$k = -a + \frac{h^2}{a}$$

$$\Rightarrow h^2 = a(a + k)$$

23. (a, c)

$$\sqrt{(x-5)^2 + (y-7)^2} + \sqrt{(x+1)^2 + (y+1)^2} = 12$$

where $P(x, y)$, $S(5, 7)$ and $S'(-1, -1)$.

$$SP + S'P = 2a \Rightarrow \text{conic is ellipse}$$

$$2ae = \sqrt{36 + 64}$$

$$2a = 12 \Rightarrow e = \frac{5}{6}$$

$$\text{Major axis } y - 7 = \frac{7+1}{5+1}(x-5)$$

$$\Rightarrow 4x - 3y + 1 = 0$$

DPP 6.2

Single Correct Answer Type

1. (d) Equation of normal to ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ is

$$\frac{x}{\cos \theta} - \frac{2y}{\sin \theta} = 1 - 4 \quad (i)$$

$$\text{Comparing with } 2x - \frac{8}{3}\lambda y = -3$$

$$\cos \theta = \frac{1}{2}, \lambda = \frac{3}{4 \sin \theta} = \frac{\sqrt{3}}{2}$$

2. (b) Tangent at point $P(a \cos \theta, b \sin \theta)$ meets x -axis at $\frac{a}{\cos \theta}$ Normal at point $P(a \cos \theta, b \sin \theta)$ meets x -axis at

$$\frac{(a^2 - b^2)}{a} \cos \theta$$

$$\text{According to the question } \frac{a}{\cos \theta} - \frac{(a^2 - b^2)}{a} \cos \theta = a$$

$$\Rightarrow e^2 \cos^2 \theta = 1 - \cos \theta \Rightarrow e = \frac{\sqrt{1 - \cos \theta}}{\cos \theta}$$

3. (d) Equation of normal to ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ at $P(3 \cos \theta, 2 \sin \theta)$ is

$$3x \sec \theta - 2y \csc \theta = 5$$

If it is tangent to circle $x^2 + y^2 = 3$, then

$$\Rightarrow \frac{5}{\sqrt{9 \sec^2 \theta + 4 \csc^2 \theta}} = \sqrt{3}$$

$$9 \sec^2 \theta + 4 \csc^2 \theta = 9 + 4 + 9 \tan^2 \theta + 4 \cot^2 \theta$$

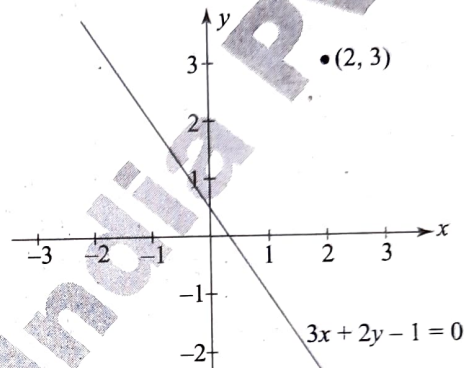
$$= 25 + (3 \tan \theta - 2 \cot \theta)^2$$

$$\therefore (9 \sec^2 \theta + 4 \csc^2 \theta)_{\min} = 25$$

 $\therefore$ no such θ exists.

Hence no tangents to circle which is normal to ellipse.

4. (d)



From the position of line and point, it is clear that no such ellipse exists.

5. (a) $x \cos \alpha + y \sin \alpha = 4$

$$\therefore y = (-\cot \alpha)x + 4 \operatorname{cosec} \alpha$$

$$\therefore m = -\cot \alpha, c = 4 \operatorname{cosec} \alpha, a^2 = 25, b^2 = 9$$

$$\text{Now } c^2 = a^2 m^2 + b^2$$

$$\therefore 16 \operatorname{cosec}^2 \alpha = 25 \cot^2 \alpha + 9$$

$$\therefore 16(1 + \cot^2 \alpha) = 25 \cot^2 \alpha + 9$$

$$\therefore 7 = 9 \cot^2 \alpha \Rightarrow \cot \alpha = \frac{\sqrt{7}}{3} \Rightarrow \tan \alpha = \frac{3}{\sqrt{7}}$$

$$\therefore \alpha = \tan^{-1}(3/\sqrt{7})$$

6. (c) The equation of the normal at $P(\theta)$ on the ellipse is

$$4x \sec \theta - 3y \csc \theta = 7$$

This meets the coordinate axes at

$$M\left(\frac{7}{4} \cos \theta, 0\right), N\left(0, -\frac{7}{3} \sin \theta\right)$$

$$\therefore PM^2 = \left(4 - \frac{7}{4}\right)^2 \cos^2 \theta + 9 \sin^2 \theta$$

$$= \frac{9}{16} (9 \cos^2 \theta + 16 \sin^2 \theta)$$

$$PN^2 = 16 \cos^2 \theta + \left(3 + \frac{7}{3}\right)^2 \sin^2 \theta$$

$$= \frac{16}{9} (9 \cos^2 \theta + 16 \sin^2 \theta)$$

$$\therefore PM^2 : PN^2 = 9^2 : 16^2$$

$$\Rightarrow |PM| : |PN| = 9 : 16$$

7. (a) At a point (x_1, y_1) on ellipse, normal will be

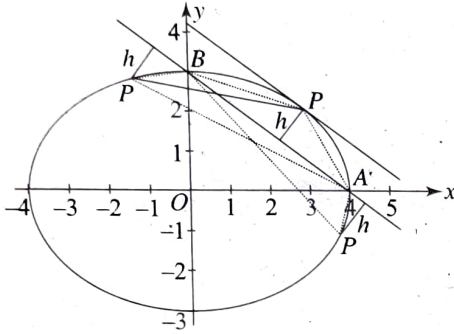
$$\frac{(x - x_1)a^2}{x_1} = \frac{(y - y_1)b^2}{y_1}$$

$$\text{At } G, y = 0 \Rightarrow x = CG = \frac{x_1(a^2 - b^2)}{a^2}$$

$$\text{At } g, x = 0 \Rightarrow y = Cg = \frac{y_1(b^2 - a^2)}{b^2}$$

$$\text{Now, } \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1 \Rightarrow a^2(CG)^2 + b^2(Cg)^2 = (a^2 - b^2)^2$$

8. (b) Put $\theta = 0^\circ$, we get rectangle formed by tangents at the extremities of major and minor axis.
 9. (d) Let the perpendicular distance of P from the line be h ,



$$\therefore \frac{1}{2} \times h \times 5 = 6(\sqrt{2} - 1)$$

$$\therefore h = \frac{12}{5}(\sqrt{2} - 1)$$

Also tangent parallel to the given line is $4y + 3x = 12\sqrt{2}$

Its distance from the line $4y + 3x = 12$ is $h = \frac{12}{5}(\sqrt{2} - 1)$

Hence there are three points as shown in the figure.

10. (a) $\frac{x^2}{5^2} + \frac{y^2}{4^2} = 1$

Any tangent to the ellipse is $\frac{x \cos \theta}{5} + \frac{y \sin \theta}{4} = 1$

This meets $x = a = 5$ at $T_1 \left\{ 5, \frac{4}{\sin \theta}(1 - \cos \theta) \right\}$
 $= \left\{ 5, 4 \tan \frac{\theta}{2} \right\}$ and meets $x = -a = -5$ at

$$T_2 \left\{ -5, \frac{4}{\sin \theta}(1 + \cos \theta) \right\} = \left\{ -5, 4 \cot \frac{\theta}{2} \right\}$$

The circle on T_1, T_2 as diameter is

$$(x - 5)(x + 5) + \left(y - 4 \tan \frac{\theta}{2} \right) \left(y - 4 \cot \frac{\theta}{2} \right) = 0$$

$$x^2 + y^2 - 4y \left(\tan \frac{\theta}{2} + \cot \frac{\theta}{2} \right) - 25 + 16 = 0$$

This is obviously satisfied by $(3, 0)$.

11. (a) It is the square of minimum distance between 2 curves

$$\frac{x^2}{16} + \frac{y^2}{9} = 1 \text{ and } y = x - 5.$$

Tangent to ellipse parallel to given line $y = x - 5$ is $y = x \pm 5$

Since $y = x - 5$ is tangent to ellipse, required distance is 0

12. (c) $S_1: C_1(-5, 12), r_1 = 16$

$$S_2: C_2(5, 12), r_2 = 4$$

$$C_1C_2 = r + 4$$

(Where C is the centre of circle touching C_2 externally and C_1 internally)

$$CC_1 = 16 - r$$

(Not $r = 16$, $\because S_2$ is contained by S_1)

$$\Rightarrow CC_1 + CC_2 = 20$$

$\therefore$ Locus of C is the ellipse with foci at C_1 and C_2 and length of major axis = 20

$\therefore$ Locus of C is

$$\frac{x^2}{100} + \frac{(y - 12)^2}{75} = 1 \quad (i)$$

According to question $y = ax$ is tangent to this ellipse from $(0, 0)$,

Equation of tangent having slope m to (i)

$$\text{is } y - 12 = mx \pm \sqrt{100m^2 + 75}$$

But this is passing through $(0, 0)$

$$\Rightarrow -12 = -\sqrt{100m^2 + 75}$$

$$\Rightarrow m^2 = \frac{69}{100}$$

$$\Rightarrow a = \pm \frac{\sqrt{13}}{10}$$

13. (a) The equations of the normals to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the points whose eccentric angles are θ and $\frac{\pi}{2} + \theta$ are

$$ax \sec \theta - by \operatorname{cosec} \theta = a^2 - b^2$$

and, $-ax \operatorname{cosec} \theta - by \sec \theta = a^2 - b^2$ respectively.

Since ω is the angle between these two normals. Therefore,

$$\tan \omega = \left| \frac{\frac{a}{b} \tan \theta + \frac{a}{b} \cot \theta}{1 - \frac{a^2}{b^2}} \right|$$

$$\Rightarrow \tan \omega = \left| \frac{ab(\tan \theta + \cot \theta)}{b^2 - a^2} \right|$$

$$\Rightarrow \tan \omega = \left| \frac{2ab}{(\sin 2\theta)(b^2 - a^2)} \right|$$

$$\Rightarrow \tan \omega = \frac{2ab}{(a^2 - b^2) \sin 2\theta}$$

$$\Rightarrow \tan \omega = \frac{2a^2 \sqrt{1 - e^2}}{a^2 e^2 \sin 2\theta}$$

$$\Rightarrow \frac{2 \cot \omega}{\sin 2\theta} = \frac{e^2}{\sqrt{1 - e^2}}$$

14. (c) Any point on $(t + 2)(x + y) = 1$ is $\left(\alpha, \frac{1}{t + 2} - \alpha \right)$

$$\text{Equation of chord of contact is } 4x\alpha + 16y \left(\frac{1}{t + 2} - \alpha \right) = 1$$

$$\Rightarrow \alpha(4x - 16y) + \left(\frac{16}{t + 2} y - 1 \right) = 0$$

The chord of contact passes through the intersection of

$$x - 4y = 0 \text{ and } \left(\frac{16}{t + 2} y - 1 \right) = 0$$

∴ The point of intersection is $\left(\frac{t+2}{4}, \frac{t+2}{16}\right)$

It lies inside the ellipse if $4\left(\frac{t+2}{4}\right)^2 + 16\left(\frac{t+2}{16}\right)^2 - 1 < 0$

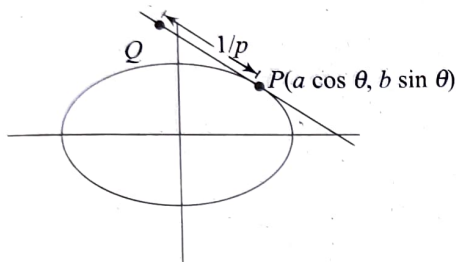
$$\Rightarrow (t+2)^2 < \frac{16}{5}$$

$$\Rightarrow \left(-\frac{4}{\sqrt{5}} - 2\right) < t < \left(\frac{4}{\sqrt{5}} - 2\right)$$

$$\Rightarrow t = -3, -1, \text{ (as } t \neq -2)$$

15. (a) Equation of the tangent at P is

$$\frac{x - a \cos \theta}{a \sin \theta} = \frac{y - b \sin \theta}{-b \cos \theta}$$



The distance of the tangent from the origin is

$$p = \frac{ab}{\sqrt{b^2 \cos^2 \theta + a^2 \sin^2 \theta}}$$

$$\Rightarrow \frac{1}{p} = \frac{\sqrt{b^2 \cos^2 \theta + a^2 \sin^2 \theta}}{ab}$$

Now the coordinates of the point Q are given as follows

$$\frac{x - a \cos \theta}{-a \sin \theta} = \frac{y - b \sin \theta}{b \cos \theta} = \frac{1}{p} = \frac{\sqrt{b^2 \cos^2 \theta + a^2 \sin^2 \theta}}{ab}$$

$$\Rightarrow x = a \cos \theta - \frac{a \sin \theta}{ab} \text{ and } y = b \sin \theta + \frac{b \cos \theta}{ab}$$

$$\Rightarrow \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 + \frac{1}{a^2 b^2} \text{ is the required locus.}$$

16. (b) P lies on $x^2 + y^2 = 5$ (director circle of given ellipse)

$$\text{Let } P(\sqrt{5} \cos \theta, \sqrt{5} \sin \theta)$$

Since $\triangle QPR$ is right angled at point P, circumcenter is mid-point of QR.

QR is chord of contact w.r.t. point P,

$$\therefore \text{ Its equation is } \frac{x\sqrt{5} \cos \theta}{4} + \frac{y\sqrt{5} \sin \theta}{1} = 1 \quad (i)$$

Also equation of chord QR which is bisected at point (h, k) is

$$\frac{xh}{h^2 + 4k^2} + \frac{4ky}{h^2 + 4k^2} = 1 \quad (\text{using } T = S_1) \quad (ii)$$

Comparing coefficients of equations (i) and (ii),

$$\frac{\sqrt{5} \cos \theta}{4} = \frac{h}{h^2 + 4k^2} \text{ and } \sqrt{5} \sin \theta = \frac{4k}{h^2 + 4k^2}$$

$$\Rightarrow x^2 + y^2 = \frac{5}{16} (x^2 + 4y^2)^2$$

17. (b) Normal at $P(a \cos \theta, b \sin \theta)$ is

$$ax \sec \theta - by \operatorname{cosec} \theta = a^2 - b^2$$

Homogenising with auxilliary circle

$$\frac{x^2 + y^2}{a^2} = \frac{(ax \sec \theta - by \operatorname{cosec} \theta)^2}{(a^2 - b^2)^2}$$

∴ For $\angle QOR = 90^\circ$

Coefficient of x^2 + Coefficient of $y^2 = 0$

$$1 - \frac{a^4 \sec^2 \theta}{(a^2 - b^2)^2} + 1 - \frac{a^2 b^2 \operatorname{cosec}^2 \theta}{(a^2 - b^2)^2} = 0$$

$$a^4 - 5a^2 b^2 + 2b^4 = a^4 \tan^2 \theta + a^2 b^2 \cot^2 \theta$$

∴ $AM \geq GM$

$$a^4 - 5a^2 b^2 + 2b^4 \geq 2a^3 b$$

Multiple Correct Answers Type

18. (a, b, c, d)

$$P = (a \cos \theta, b \sin \theta) \text{ and } Q = (-a \sin \theta, b \cos \theta)$$

$$\text{Tangent at P, } \frac{x \cos \theta}{a} + \frac{y \sin \theta}{b} = 1 \quad (1)$$

$$\frac{-x \sin \theta}{a} + \frac{y \cos \theta}{b} = 1 \quad (2)$$

Elimination θ , by squaring and adding

$$(1)^2 + (2)^2 \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 2, \text{ which is required locus.}$$

∴ (a) is correct

$$\text{Now mid}(PQ) = \left(\frac{a(\cos \theta - \sin \theta)}{2}, \frac{b(\sin \theta + \cos \theta)}{2} \right) = (x, y)$$

$$\cos \theta - \sin \theta = \frac{2x}{a} \quad (3)$$

$$\cos \theta + \sin \theta = \frac{2y}{b} \quad (4)$$

$$\text{Squaring and adding (3) and (4), we get locus as } \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{2}$$

∴ (b) is correct

$$\text{Slope of } OP = \frac{b \sin \theta}{a \cos \theta} = m_1$$

$$\text{Slope of } OQ = \frac{-b \cos \theta}{a \sin \theta} = m_2$$

$$\therefore m_1 m_2 = \frac{-b^2}{a^2}$$

∴ (c) is correct

$$\text{Now area of triangle } OPQ = \frac{1}{2} \begin{vmatrix} a \cos \theta & b \sin \theta \\ -a \sin \theta & b \cos \theta \end{vmatrix}$$

$$= \frac{1}{2} ab(\cos^2 \theta + \sin^2 \theta) = \frac{1}{2} ab$$

∴ (d) is correct

19. (b, c, d)

Without loss of generality assume $a > b$

Foci of 1st ellipse are $(\pm ae, 0)$

Putting this point in $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, we get

$$\frac{a^2 e^2}{b^2} = 1$$

The above quantity may be negative or positive, hence option (a) is not correct

Auxiliary circle for $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $x^2 + y^2 = a^2$

for $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ is $x^2 + y^2 = a^2$

Director circle for $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ is $x^2 + y^2 = a^2 + b^2$

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \text{ is}$$

Area of the ellipses $= \pi ab$

20. (a, b)

Let P and Q be the points (α, β) and (α_1, β_1)

$\Rightarrow$ Equations of AB and CD are $\frac{x}{a}\alpha + \frac{y}{b}\beta = 1$ and $\frac{x}{a}\alpha_1 + \frac{y}{b}\beta_1 = 1$
(Chord of contact)

These lines are parallel

$$\Rightarrow \frac{\alpha}{\alpha_1} = \frac{\beta}{\beta_1} = k$$

$$\text{Also } \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} = \frac{\alpha_1^2}{a^2} + \frac{\beta_1^2}{b^2}$$

$$\Rightarrow \frac{\alpha}{\alpha_1} = \frac{\beta}{\beta_1} = -1$$

$\Rightarrow PQ$ passes through origin and is bisected at the origin.

For the required condition, $(2\alpha, \alpha)$ should lie inside the circle and outside the ellipse

$$\text{i.e., } (2\alpha)^2 + 3\alpha^2 - 9 > 0 \text{ and } (2\alpha)^2 + \alpha^2 - 12 < 0$$

$$\text{i.e., } \frac{9}{7} < \alpha^2 < \frac{12}{5}$$

4. (d) $2ae = SS' = 1$

$p_1 p_2 = b^2$, where p_1 and p_2 are the length of perpendicular from S and S' to the tangent

$$\frac{5}{\sqrt{2}} \cdot \frac{4}{\sqrt{2}} = b^2$$

$$\Rightarrow b^2 = 10$$

$$\Rightarrow b^2 = 10 = a^2 - e^2 a^2$$

$$\Rightarrow a^2 = \frac{41}{4}$$

5. (a)

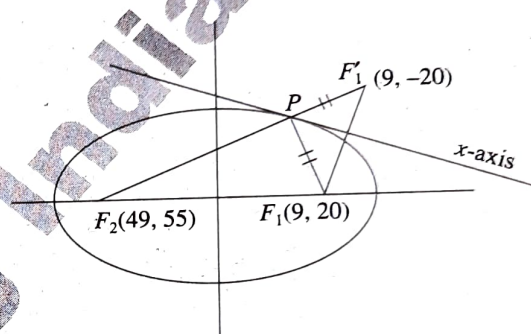
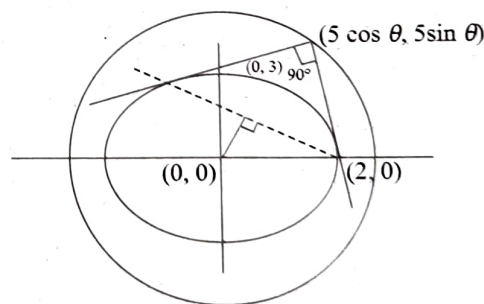


Image of F_1 in tangent (x -axis) is F_1' . From the property, points F_2, P, F_1' are collinear.

$$2a = F_1 P + F_2 P = F_1' P + F_2 P = F_1' F_2 = 85$$

6. (c)



Perpendicular tangents meet on director circle $x^2 + y^2 = 25$,

Any point on this circle is $P(5 \cos \theta, 5 \sin \theta)$

Equation of chord of contact of ellipse w.r.t. this point is

$$\frac{5 \cos \theta x}{16} + \frac{5 \sin \theta y}{9} = 1$$

$$\text{Distance from center} = \frac{1}{\sqrt{\frac{25}{256} \cos^2 \theta + \frac{25}{81} \sin^2 \theta}}$$

For maximum value $\cos \theta = 1$ and $\sin \theta = 0$

$$\text{Maximum distance} = \frac{16}{5}$$

DPP 6.3

Single Correct Answer Type

1. (d) Given circle is director circle, so angle between tangents is $\frac{\pi}{2}$.

$$2. (c) \frac{(3x+4y-2)^2}{100} + \frac{(4x-3y+5)^2}{625} = 1$$

$$\Rightarrow \frac{\left(\frac{3x+4y-2}{5}\right)^2}{4} + \frac{\left(\frac{4x-3y+5}{5}\right)^2}{25} = 1$$

$$\Rightarrow a^2 = 4 \text{ and } b^2 = 25$$

$$\text{Radius of director circle} = \sqrt{a^2 + b^2} = \sqrt{29}$$

3. (b) The given curve is $\frac{x^2}{9} + \frac{y^2}{3} = 1$, whose director circle is

$$x^2 + y^2 = 12.$$

7. (c) Let equation of tangent $y = mx + c$

$$P = \frac{|c|}{\sqrt{1+m^2}} \quad P_1 = \frac{|c+ae^2m|}{\sqrt{1+m^2}}$$

$$P_2 = \frac{|c-ae^2m|}{\sqrt{1+m^2}} \quad P_3 = \frac{|c+am|}{\sqrt{1+m^2}}$$

$$P_4 = \frac{|c-am|}{\sqrt{1+m^2}}$$

$$\text{So, } \frac{P_1 P_2 - P^2}{P_3 P_4 - P^2} = \frac{c^2 - a^2 e^2 m^2 - c^2}{c^2 - a^2 m^2 - c^2} = e^2$$

8. (d) Let $\theta = \cos^{-1}\left(\frac{-1}{\sqrt{5}}\right) \Rightarrow \cos \theta = \frac{-1}{\sqrt{5}} \Rightarrow \tan \theta = -2$

Foci are $(\pm 5, 0)$

Equation of line through $(-5, 0)$ with slope -2 is $y = -2(x+5)$

or $y = -2x - 10$

This line meets the ellipse above X -axis at $(-6, 2)$

Reflected ray passes through the other focus $(5, 0)$

$$\therefore \text{Slope} = \frac{2-0}{-6-5} = -\frac{2}{11}$$

9. (c) SM' and $S'M$ intersect at mid-point of PG (where G is point of intersection of normal at P with major axis)

$$G\left(\frac{8\sqrt{2}}{5}, 0\right)$$

$$\text{Mid-point of } PG \left(\frac{41}{10\sqrt{2}}, \frac{3}{2\sqrt{2}}\right)$$

10. (b) Let the equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Equation of a normal at any point $P(\theta)$ on the ellipse is

$$(a \sec \theta)x - (b \operatorname{cosec} \theta)y = a^2 e^2$$

Let normal is passing through a point $A(h, k)$

$$\Rightarrow (ah \sec \theta - a^2 e^2)^2 = (bk \operatorname{cosec} \theta)^2$$

$$\Rightarrow a^2 h^2 \sec^2 \theta - 2a^3 e^2 h \sec \theta + a^4 e^4$$

$$= b^2 k^2 \operatorname{cosec}^2 \theta = b^2 k^2 \left(\frac{\sec^2 \theta}{\sec^2 \theta - 1} \right)$$

$$\Rightarrow a^2 h^2 \sec^4 \theta - 2a^3 e^2 h \sec^3 \theta + (a^4 e^4 - a^2 h^2 - b^2 k^2) \sec^2 \theta + 2a^3 e^2 h \sec \theta - a^4 e^4 = 0 \quad (1)$$

If α, β, γ and δ be the roots of the above equation, then

$$\Sigma \sec \alpha = \frac{2a^3 e^2 h}{a^2 h^2} = \frac{2ae^2}{h}$$

Multiplying equation (1) by $\cos^4 \theta$, it reduces to

$$a^4 e^4 \cos^4 \theta - 2a^3 e^2 h \cos^3 \theta - (a^4 e^4 - a^2 h^2 - b^2 k^2) \cos^2 \theta + 2a^3 e^2 h \cos \theta - a^4 e^4 = 0 \quad (2)$$

$$\text{Then } \Sigma \cos \alpha = \frac{2a^3 e^2 h}{a^4 e^4} = \frac{2h}{ae^2}$$

Hence, we have

$$(\Sigma \sec \alpha)(\Sigma \cos \alpha) = \frac{2ae^2}{h} \cdot \frac{2h}{ae^2} = 4$$

11. [c] We know that locus of the point of intersection of perpendicular tangents to the given ellipse is $x^2 + y^2 = a^2 + b^2$. Any point on this circle can be taken as

$$P(\sqrt{a^2 + b^2} \cos \theta, \sqrt{a^2 + b^2} \sin \theta)$$

The equation of the chord of contact of tangents from P is

$$\frac{x}{a^2} \sqrt{a^2 + b^2} \cos \theta + \frac{y}{b^2} \sqrt{a^2 + b^2} \sin \theta = 1.$$

Let this line be a tangent to the fixed ellipse $\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$.

$$\Rightarrow \frac{x}{A} \cos \theta + \frac{y}{B} \sin \theta = 1,$$

$$\text{Where } A = \frac{a^2}{\sqrt{a^2 + b^2}}, B = \frac{b^2}{\sqrt{a^2 + b^2}}$$

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{(a^2 + b^2)}.$$

12. (c)

$$(p) (CA)(CT) = (e^2 a \cos \theta) \left(\frac{a^2}{a \cos \theta} \right) = a^2 - b^2 = 16$$

$$(q) (PN)(PB) = a^2 = 25$$

$$(r) (S_1 N_1)(S_2 N_2) = b^2 = 9$$

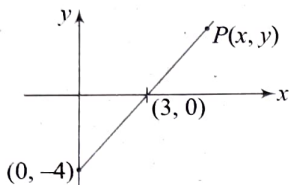
$$(s) (S_1 P)(S_2 P) = [a + e(a \cos \theta)][a - e(a \cos \theta)] = a^2 \sin^2 \theta + b^2 \cos^2 \theta = 17$$

CHAPTER 7

DPP 7.1

Single Correct Answer Type

1. (d) We have $\sqrt{x^2 + (y+4)^2} - \sqrt{(x-3)^2 + y^2} = 5$. The distance between the two points (3, 0) and (0, -4) is 5.



Locus of P is the part of line $\frac{x}{3} - \frac{y}{4} = 1$.

2. (b) $x^2 - y^2 = 4$ is rectangular hyperbola.

$$\therefore e = \sqrt{2}$$

$$\text{Required distance} = ae - a/e = 2\sqrt{2} - 2/\sqrt{2} = \sqrt{2}$$

3. (d) For hyperbola $\frac{x^2}{36} - \frac{y^2}{k^2} = 1$, $a = 6$

$\therefore$ Abscissa of any point on hyperbola must be ≥ 6

$\therefore$ (10, 4) can lie on hyperbola

4. (a) For ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$, $a = 5$ and $b = 4$

$$\text{For hyperbola } \frac{x^2}{25} - \frac{y^2}{16} = 1, a = 5 \text{ and } b = 4$$

$$\text{For ellipse } a^2e^2 = a^2 - b^2 = 25 - 16 = 9$$

$$\text{For hyperbola } a^2e^2 = a^2 + b^2 = 25 + 16 = 41$$

$\therefore$ center and vertices only are common.

5. (b) Distance between foci = 8

$$\therefore 2ae = 8 \text{ also } e = 2; \therefore 2a = 4$$

$$\Rightarrow a = 2 \Rightarrow a^2 = 4 \therefore b^2 = 4(4 - 1) = 12$$

$$\therefore \text{Equation of hyperbola is } \frac{x^2}{4} - \frac{y^2}{12} = 1.$$

6. (c) Centre (0, 0), vertex (4, 0)

$$\Rightarrow a = 4 \text{ and focus } (6, 0)$$

$$\Rightarrow ae = 6 \Rightarrow e = \frac{3}{2}. \text{ Therefore } b = \sqrt{20}$$

$$\text{Hence required equation is } \frac{x^2}{16} - \frac{y^2}{20} = 1$$

$$\text{i.e., } 5x^2 - 4y^2 = 80$$

7. (c) $(9 - \lambda)(4 - \lambda) < 0 \Rightarrow 4 < \lambda < 9$

$$\Rightarrow b - a = 5$$

8. (a) For $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $e^2 = 1 + \frac{b^2}{a^2} = \frac{a^2 + b^2}{a^2}$

$$\text{For } \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1, (e')^2 = 1 + \frac{a^2}{b^2} = \frac{b^2 + a^2}{b^2}$$

$$\Rightarrow \frac{1}{e^2} + \frac{1}{(e')^2}$$

$$= \frac{a^2}{a^2 + b^2} + \frac{b^2}{b^2 + a^2}$$

$$= \frac{a^2 + b^2}{a^2 + b^2} = 1$$

So the point $\left(\frac{1}{e}, \frac{1}{e'}\right)$ lie on $x^2 + y^2 = 1$

9. (b) Foci of hyperbola are $(\pm\sqrt{41}, 0)$

$\therefore P$ lies on the circle $x^2 + y^2 = 41$

Any point on hyperbola is $(4 \sec \theta, 5 \tan \theta)$

$$\Rightarrow 16 \sec^2 \theta + 25 \tan^2 \theta = 41$$

$$\Rightarrow \tan \theta = \frac{5}{\sqrt{41}} \text{ and } \sec \theta = \frac{\sqrt{66}}{\sqrt{41}}$$

$$\therefore PF_1 + PF_2 = e \left(4 \sec \theta - \frac{a}{e} \right) + e \left(4 \sec \theta + \frac{a}{e} \right) \\ = 8 \operatorname{sech} \theta = 2\sqrt{66}$$

10. (a) Given hyperbola is $\frac{(x-1)^2}{16} - \frac{(y+1)^2}{9} = 1$

Length of the transverse axis is $2a = 8$.

11. (c) Distance between directrices = $\frac{2a}{e} = 4$

$$\Rightarrow a = 2e.$$

P is nearest to $3x + 4y - 10 = 0$

Distance of P from this directrix is

$$\Rightarrow ae - \frac{a}{e} = 8$$

$$\Rightarrow e = \sqrt{5}, a = 2\sqrt{5}$$

$$\Rightarrow CP = ae = 10$$

12. (a) Foci of $\frac{x^2}{16} + \frac{y^2}{4} = 1$ are $(\pm\sqrt{12}, 0)$

$$\text{Foci of } \frac{x^2}{a^2} - \frac{y^2}{3} = 1 \text{ are } (\pm\sqrt{a^2 + 3}, 0)$$

$$\text{Given } a^2 + 3 = 12 \Rightarrow a^2 = 9 \Rightarrow a = 3$$

13. (a) The difference between the focal distances is constant for a hyperbola.

For a rectangular hyperbola latus rectum = transverse axis.

One of the foci is $S(2, 0)$. Let another focus is $S'(h, k)$

$P(0, 0)$ is point on the hyperbola

$$\therefore |S'P - SP| = |\sqrt{h^2 + k^2} - 2| = 4$$

$$\Rightarrow \sqrt{h^2 + k^2} = 6 \Rightarrow h^2 + k^2 = 36$$

$\therefore$ Locus of (h, k) is $x^2 + y^2 = 36$

14. (a) A and B are $(\sqrt{5}, 1)$ and $(-\sqrt{5}, -1)$.

Let C be $(2 \sec \theta, 2 \tan \theta)$

O(0, 0) is the mid point of AB

Slope of OC = $\sin \theta$ and slope of AB = $\frac{1}{\sqrt{5}}$

Since $OC \perp AB$

So, $\sin \theta = -\sqrt{5}$ which is impossible.

15. (a) Let $(3 \tan(\theta + 60^\circ), 2 \tan(\theta + 30^\circ)) = (h, k)$

$$\therefore \tan(\theta + 60^\circ) = \frac{h}{3} \quad (1)$$

$$\text{and } \tan(\theta + 30^\circ) = \frac{k}{2} \quad (2)$$

$$\tan 30^\circ = \tan[(\theta + 60^\circ) - (\theta + 30^\circ)]$$

$$\Rightarrow \frac{1}{\sqrt{3}} = \frac{\tan(\theta + 60^\circ) - \tan(\theta + 30^\circ)}{1 + \tan(\theta + 60^\circ) \tan(\theta + 30^\circ)}$$

$$\Rightarrow \frac{1}{\sqrt{3}} = \frac{\frac{x}{3} - \frac{y}{2}}{1 + \frac{xy}{6}}$$

$$\Rightarrow xy - 2\sqrt{3}x + 3\sqrt{3}y + 6 = 0$$

$$\Rightarrow (x + 3\sqrt{3})(y - 2\sqrt{3}) + 24 = 0$$

$$\Rightarrow \text{center is } (-3\sqrt{3}, 2\sqrt{3})$$

Multiple Correct Answers Type

16. (b, c)

Focus of ellipse lying to the right of the y-axis is $S_1(ae, 0)$

Focus of hyperbola lying to the right of the y-axis is $S_2(aE/2, 0)$

Now $P(\alpha, \beta)$ is equidistance from S_1 and S_2

$$\therefore S_1P = S_2P \Rightarrow a - e\alpha = E\alpha - \left(\frac{a}{2}\right) \quad (1)$$

Also P lies on perpendicular bisector of S_1S_2

$$\therefore \alpha = \frac{ae + \frac{a}{2}E}{2} \quad (2)$$

$$\text{From (1) and (2), } E^2 + 3eE + (2e^2 - 6) = 0$$

$$\Rightarrow E = \frac{\sqrt{e^2 + 24} - 3e}{2}$$

17. (b, c, d)

$$\text{Ellipse is } \frac{x^2}{4} + \frac{y^2}{3} = 1$$

$$\text{Here } \frac{3}{4} = 1 - e^2 \Rightarrow e = \frac{1}{2}$$

Foci are $(\pm 1, 0)$

Now the hyperbola is having the same focus, i.e. $(\pm 1, 0)$.

Let e_1 be the eccentricity of hyperbola

$$2ae_1 = 2$$

$$\text{But } 2a = \frac{1}{2} \text{ So, } e_1 = 4$$

$$b^2 = a^2(e_1^2 - 1) = \frac{1}{16}(16 - 1) = \frac{15}{16}$$

So, the equation of the hyperbola is

$$\frac{x^2}{\frac{1}{16}} - \frac{y^2}{\frac{15}{16}} = 1 \Rightarrow \frac{x^2}{1} - \frac{y^2}{15} = \frac{1}{16}$$

$$\text{Its distance between the directrices} = \frac{2a}{e_1} = \frac{1}{2 \times 4} = \frac{1}{8} \text{ units}$$

$$\text{Length of latus-rectum} = \frac{2b^2}{a} = \frac{2 \times 15 \times 4}{16 \times 1} = \frac{15}{2} \text{ units}$$

18. (a, b, c, d)

$$(a) \sqrt{x} + \sqrt{y} = 1$$

$$\Rightarrow x + y + 2\sqrt{xy} = 1$$

$$\Rightarrow 4xy = (1 - x - y)^2$$

$$(b) \frac{1}{x} + \frac{1}{y} = 1 \Rightarrow xy - x - y = 0 \text{ is a hyperbola}$$

$$(c) \frac{1}{x} + \frac{1}{y} + \frac{1}{x+y} = 0$$

$$\Rightarrow x^2 + 3xy + y^2 = 0, \text{ which is a pair of lines.}$$

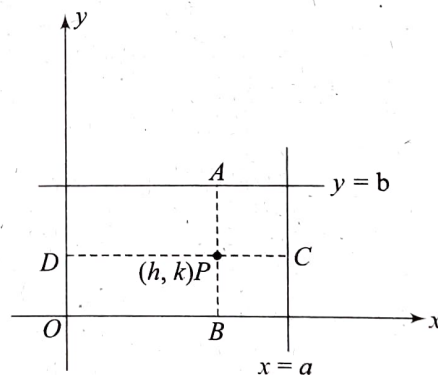
$$(d) x + y - \frac{1}{x+y} = 0$$

$$\Rightarrow (x + y)^2 = 1$$

$$\Rightarrow x + y = \pm 1 \text{ which is a pair of lines.}$$

19. (c, d)

Let the two sides of the rectangle lie along x-axis and y-axis.



Let point P be (h, k)

Given that

$$(PA)^2 + (PB)^2 = 2(PC^2 + PD^2)$$

$$\Rightarrow k^2 + (k - b)^2 = 2(h^2 + (a - h)^2)$$

$$\Rightarrow 2k^2 - 2kb + b^2 = 4h^2 - 4ah + 2a^2$$

Replacing h by x and k by y

$$\Rightarrow 2y^2 - 2by + b^2 = 4x^2 - 4ax + 2a^2$$

$$\text{or } 2(y^2 - by) + b^2 = 4(x^2 - ax) + 2a^2$$

S.44 Coordinate Geometry

$$\text{or } 4\left(x - \frac{a}{2}\right)^2 - 2\left(y - \frac{b}{2}\right)^2 = \frac{b^2}{2} - a^2$$

Hence it is a hyperbola if $\frac{b^2}{2} - a^2 \neq 0$

or pair of lines if $\frac{b^2}{2} - a^2 = 0$

20. (a, b)

$$\text{Consider } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Let one of the vertices be $(a, 0)$

Foci are $(\pm ae, 0)$

According to the question we have

$$ae - a = 1 \text{ and } ae + a = 3$$

Solving we get $a = 1$ and $e = 2$

$$\therefore e^2 = 1 + \frac{b^2}{a^2} \Rightarrow b^2 = 3$$

$$\therefore \text{Equation of hyperbola is } \frac{x^2}{1} - \frac{y^2}{3} = 1 \text{ or } 3x^2 - y^2 = 3$$

$$\text{For } \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1 \text{ is } e^2 = 1 + \frac{a^2}{b^2} \Rightarrow b^2 = \frac{1}{3}$$

$$\therefore \text{Hyperbola will be } x^2 - 3y^2 + 3 = 0$$

$$3. (b) \text{ Tangent to } y^2 = 8x \text{ is } y = mx + \frac{2}{m}$$

$$\text{Tangent to } \frac{x^2}{1} - \frac{y^2}{3} = 1 \text{ is } y = mx \pm \sqrt{m^2 - 3}$$

on comparing, we get

$$m = \pm 2$$

$$4. (a) \text{ Any point of } xy = 8 \text{ is } P\left(\sqrt{8t}, \frac{\sqrt{8}}{t}\right)$$

$$\therefore \text{Equation of the tangent at } P \text{ is } \frac{x}{\frac{16t}{\sqrt{8}}} + \frac{y}{\frac{16}{\sqrt{8t}}} = 1$$

$$\therefore \text{Area of the triangle} = \frac{1}{2} \cdot \frac{16t}{\sqrt{8}} \cdot \frac{16}{\sqrt{8t}} = 16$$

$$5. (c) \text{ Equation of tangent at point } P(\theta) \text{ is}$$

$$\frac{\sec \theta}{a}x - \frac{\tan \theta}{b}y = 1$$

$$\therefore A(0, -b \cot \theta)$$

Equation of normal at point $P(\theta)$ is

$$a \cos \theta x + b \cot \theta y = a^2 + b^2$$

$$\therefore B\left(0, \frac{a^2 + b^2}{b \cot \theta}\right)$$

Equation of circle as AB as a diameter is

$$(x-0)(x-0) + (y+b \cot \theta)\left(y - \frac{a^2 + b^2}{b \cot \theta}\right) = 0$$

$$\text{or } x^2 + (y + b \cot \theta)\left(y - \frac{a^2 + b^2}{b \cot \theta}\right) = 0$$

Clearly this passes through foci $(ae, 0)$

$$6. (d) \text{ Slope of 1st curve } \left(\frac{dy}{dx}\right)_I = -\frac{4x}{py}$$

$$\text{Slope of 2nd curve } \left(\frac{dy}{dx}\right)_{II} = \frac{x}{4y}$$

$$\text{For orthogonal intersection } \left(-\frac{4x}{py}\right)\left(\frac{x}{4y}\right) = -1$$

$$\Rightarrow x^2 = py^2$$

$$\therefore \text{From 1st curve } 5x^2 = 45$$

$$\therefore x = 3$$

$$\therefore y = 1$$

$$\therefore p(1) = (3)^2 = 9 \Rightarrow p = 9.$$

$$7. (c) \text{ The point } P\left(\frac{\pi}{6}\right) \text{ is } \left(a \sec \frac{\pi}{6}, b \tan \frac{\pi}{6}\right), \text{ i.e., } P\left(\frac{2a}{\sqrt{3}}, \frac{b}{\sqrt{3}}\right)$$

$$\therefore \text{Equation of tangent at } P \text{ is } \frac{x}{\frac{2a}{\sqrt{3}}} - \frac{y}{\frac{b}{\sqrt{3}}} = 1$$

DPP 7.2

Single Correct Answer Type

$$1. (b) 3x^2 - y^2 = 3, \frac{x^2}{1} - \frac{y^2}{3} = 1$$

Equation of tangent in terms of slope.

$$y = mx \pm \sqrt{(m^2 - 3)}$$

Here, $m = 2$,

$$\text{then } y = 2x \pm 1$$

$$2. (a) y = \frac{x+9}{x+5} = 1 + \frac{4}{x+5}$$

$$\frac{dy}{dx} \text{ at } (x_1, y_1) = \frac{-4}{(x_1+5)^2}$$

$\therefore$ Equation of tangent

$$y - y_1 = \frac{-4}{(x_1+5)^2}(x - x_1)$$

$$y - 1 = \frac{-4}{x_1+5} \cdot \frac{-4}{(x_1+5)^2} \cdot (x - x_1)$$

Since it passes through $(0, 0)$

$$(x_1+5)^2 + 4(x_1+5) + 4x_1 = 0$$

$$x_1 = -15 \text{ or } x_1 = -3. \text{ So equations are } x + 25y = 0 \text{ or } x + y = 0.$$

$$\therefore \text{Area of the triangle} = \frac{1}{2} \times \frac{\sqrt{3}a}{2} \times \sqrt{3}b = 3a^2$$

$$\therefore \frac{b}{a} = 4$$

$$\therefore e^2 = 1 + \frac{b^2}{a^2} = 17$$

8. (a) For hyperbola $\frac{x^2}{A^2} - \frac{y^2}{B^2} = 1$ equation of tangent having slope m is

$$\Rightarrow y = mx \pm \sqrt{A^2 m^2 - B^2}$$

$$\text{where } A^2 m^2 - B^2 \geq 0$$

$$\Rightarrow m^2 \geq \frac{B^2}{A^2}$$

$$\Rightarrow m^2 \geq \frac{a^3 - b^3}{a^2 - b^2}$$

$$\Rightarrow (a+b)m^2 \geq a^2 + ab + b^2$$

$$\Rightarrow (a+b)m^2 + ab \geq (a+b)^2$$

9. (c) Tangents having slope '2', $y = 2x \pm \sqrt{100 - 9} = 2x \pm \sqrt{91}$

$$\text{It meets the axis at } A \equiv \left(\mp \frac{\sqrt{91}}{2}, 0 \right), B(0, \pm \sqrt{91})$$

$$\text{Tangents having slope 'm', } y = mx \pm \sqrt{25m^2 - 9}$$

$$\text{It meets the axis at } D \left(\pm \frac{\sqrt{25m^2 - 9}}{m}, 0 \right), C \equiv (0, \mp \sqrt{25m^2 - 9})$$

ABCD is cyclic quadrilateral

$$\Rightarrow OA \times OD = OB \times OC$$

$$\Rightarrow \frac{\sqrt{91}(\sqrt{25m^2 - 9})}{2m} = \sqrt{91}(\sqrt{25m^2 - 9})$$

$$\Rightarrow 2m = 1 \Rightarrow m = \frac{1}{2}$$

10. (b) $S = 25x^2 - 16y^2 - 400 = 0$

Equation of required chord is $S_1 = T$

$$\text{Here, } S_1 = 25(5)^2 - 16(3)^2 - 400$$

$$= 625 - 144 - 400 = 81$$

$$\text{and } T = 25xx_1 - 16yy_1 - 400, \text{ where } x_1 = 5, y_1 = 3$$

$$= 25(x)(5) - 16(y)(3) - 400 = 125x - 48y - 400$$

So from (i), required chord is

$$125x - 48y - 400 = 81$$

$$\text{or } 125x - 48y = 481$$

11. (b) Slope of chord joining P and Q = slope of normal at P

$$\therefore \frac{\tan \alpha - \tan \theta}{\sec \alpha - \sec \theta} = -\frac{\tan \theta}{\sec \theta}$$

$$\therefore \tan \alpha - \tan \theta = -k \tan \theta \text{ and } \sec \alpha - \sec \theta = k \sec \theta$$

$$(1+k) \sec \theta = \sec \alpha \quad (1)$$

$$\therefore (1-k) \tan \theta = \tan \alpha \quad (2)$$

$$[(1+k) \sec \theta]^2 - [(1-k) \tan \theta]^2 = \sec^2 \alpha - \tan^2 \alpha = 1$$

$$\Rightarrow k = -2(\sec^2 \theta + \tan^2 \theta) = -4 \sec^2 \theta + 2$$

$$\text{From (2), } \tan \alpha = \tan \theta (1 + 4 \sec^2 \theta - 2) = \tan \theta (4 \sec^2 \theta - 1)$$

12. (c) Normal to hyperbola $xy = c^2$ at $\left(ct, \frac{c}{t} \right)$ is $y - \frac{c}{t} = t^2(x - ct)$

Solving with $xy = -c^2$, we get

$$\Rightarrow x \left\{ \frac{c}{t} + t^2(x - ct) \right\} + c^2 = 0$$

$$\Rightarrow t^2 x^2 + \left(\frac{c}{t} - ct^3 \right) x + c^2 = 0$$

Since line touches the curve, above equation has equal roots

$$\therefore \Delta = 0, \left(\frac{1}{t} - t^3 \right)^2 - 4t^2 = 0$$

$$\Rightarrow (1 - t^4)^2 - 4t^4 = 0$$

$$\Rightarrow t^4 - 2t^2 - 1 = 0 \text{ or } t^4 + 2t^2 - 1 = 0$$

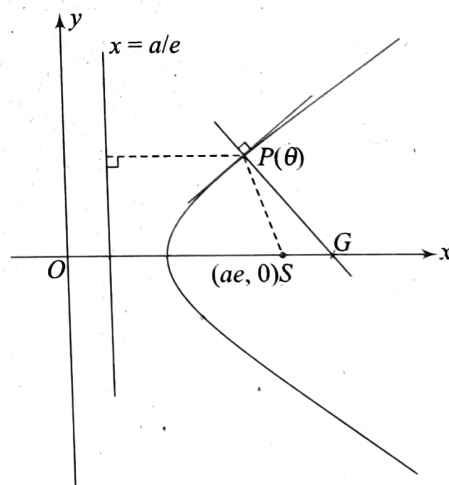
$$\Rightarrow t^2 = \frac{2 \pm \sqrt{8}}{2} \text{ or } \frac{-2 \pm \sqrt{8}}{2}$$

$$\Rightarrow t^2 = 1 + \sqrt{2} \text{ or } -1 + \sqrt{2}$$

Thus four such values of t are possible.

13. (c) Equation of normal at point $P(a \sec \theta, b \tan \theta)$ on hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ is}$$



$$a \cos \theta x + b \cot \theta y = a^2 + b^2$$

$$\therefore G \left(\frac{a^2 + b^2}{a \cos \theta}, 0 \right)$$

$$S(ae, 0)$$

$$SP = e(a \sec \theta) - a$$

$$\text{Also } SG = \frac{a^2 + b^2}{a \cos \theta} - ae = \frac{a^2 e^2}{a \cos \theta} - ae = e[e(a \sec \theta) - a]$$

$$\Rightarrow SG/SP = e = 6$$

14. (a) Equation of the normal is $ax \cos \theta + by \cot \theta = a^2 + b^2$

The normal at P meets the coordinate axes at $G \left(\frac{a^2 + b^2}{a} \sec \theta, 0 \right)$

and $g\left(0, \frac{a^2+b^2}{b} \tan \theta\right)$

$$\therefore PG^2 = \left(\frac{a^2+b^2}{a} \sec \theta - a \sec \theta\right)^2 + (b \tan \theta - 0)^2$$

$$PG^2 = \frac{b^2}{a^2} (b^2 \sec^2 \theta + a^2 \tan^2 \theta)$$

When $\tan \theta = 0$

$$PG = \frac{b^2}{a}$$

15. (a) Given hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Differentiate w.r.t. x , we get $\frac{2x}{a^2} - \frac{2yy'}{b^2} = 0$ or $y' = \frac{b^2 x}{a^2 y}$

Slope of normal at $\left(ae, \frac{b^2}{a}\right)$ is $y' = -\frac{a^2 \frac{b^2}{a}}{b^2 ae} = -\frac{1}{e}$

Now line which meets hyperbola only once has slope $\pm \frac{b}{a}$
(parallel to asymptote)

According to question $-\frac{b}{a} = -\frac{1}{e}$

$$\Rightarrow \frac{b^2}{a^2} = \frac{1}{e^2}$$

$$\Rightarrow e^2 - 1 = \frac{1}{e^2}$$

$$\Rightarrow e^4 - e^2 - 1 = 0$$

$$\Rightarrow e^2 = \frac{1 \pm \sqrt{5}}{2}$$

$$\Rightarrow e = \sqrt{\frac{1 + \sqrt{5}}{2}}$$

16. (a) Let (x_1, y_1) be point of intersection

$$\Rightarrow y_1^2 = 4ax_1, x_1 y_1 = c^2$$

$$y^2 = 4ax \quad xy = c^2$$

$$m_{T(x_1, y_1)} = \frac{2a}{y_1} = \tan \phi$$

$$M_{T(x_1, y_1)} = -\frac{y_1}{x_1} = \tan \theta$$

So $\frac{\tan \theta}{\tan \phi} = \frac{-y_1^2}{2ax_1} = -\frac{4ax_1}{2ax_1} = -2$

So $\theta = \tan^{-1}(-2 \tan \phi)$

Multiple Correct Answers Type

17. (a, b, c)

Given $B(-2, -2)$ and $C(6, 2/3)$ on hyperbola

Let $A(2t, 2/t)$

Differentiating curve $xy' + y = 0$ or $y' = -y/x = -1/t^2$

Slope of normal at A is t^2

Normal is parallel to BC , which has slope $\frac{\frac{2}{3} + 2}{6 + 2} = \frac{1}{3} = t^2$

$$\therefore t = \frac{1}{\sqrt{3}}$$

$$\therefore A\left(\frac{2}{\sqrt{3}}, 2\sqrt{3}\right)$$

Also (slope of AB) $\times$ (slope of AC) = -1

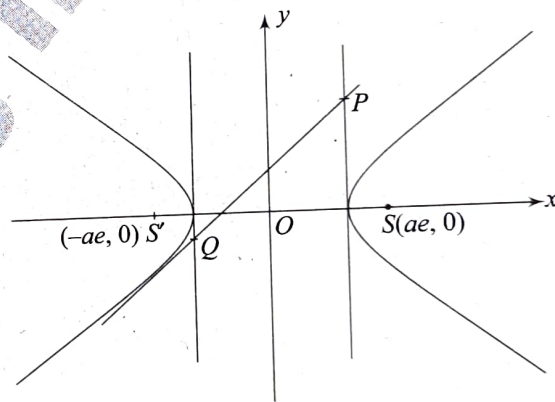
$\therefore \triangle ABC$ is a right angled $\triangle$.

Circum-center of $\triangle ABC$ is mid-point BC , $(2, -2/3)$

Circum-circle of $\triangle ABC$ is circle on BC as diameter.

Orthocenter of $\triangle ABC$ is at A .

18. (a, b)



Any tangent to the hyperbola is $\frac{x \sec \theta}{a} - \frac{y \tan \theta}{b} = 1$

Solving this line with the lines $x = \pm a$, we get the coordinates of points P and Q as

$$\left(a, b \tan \frac{\theta}{2}\right) \text{ and } \left(-a, -b \cot \frac{\theta}{2}\right).$$

Now slopes of the lines PS and QS are

$$m_{PS} = \frac{b \tan \frac{\theta}{2}}{a(1-e)}, m_{QS} = \frac{-b \cot \frac{\theta}{2}}{-a(1+e)}$$

$$\Rightarrow m_{PS} \cdot m_{QS} = \frac{-b^2}{a^2(e^2-1)} = -1$$

Similarly $m_{PS} \cdot m_{QS'} = -1$

$\Rightarrow$ line PQ subtends an angle of $\frac{\pi}{2}$ at S and S'

$\Rightarrow$ points P, Q, S and S' are concyclic.

$\Rightarrow PQ$ is diameter.

DPP 7.3

Single Correct Answer Type

1. (d) $\frac{x^2}{\cos^2 \alpha} - \frac{y^2}{\sin^2 \alpha} = 1$

Locus of perpendicular tangents is director circle,

$$x^2 + y^2 = a^2 - b^2$$

or $x^2 + y^2 = \cos^2 \alpha - \sin^2 \alpha = \cos 2\alpha$

But $0 < \alpha < \frac{\pi}{4}$

$$0 < 2\alpha < \frac{\pi}{2}$$

So there are infinite points.

2. (c) $\theta = 2 \tan^{-1} \frac{b}{a} \Rightarrow \tan \frac{\theta}{2} = \frac{b}{a}$

$$\Rightarrow \cos \frac{\theta}{2} = \frac{a}{\sqrt{a^2 + b^2}} = \frac{1}{\sqrt{1 + \frac{b^2}{a^2}}} = \frac{1}{e}$$

3. (a) The asymptotes are $\frac{x}{5} + \frac{y}{3} = 0$ and $\frac{x}{5} - \frac{y}{3} = 0$

The equation of hyperbola is $\frac{x^2}{25} - \frac{y^2}{9} = 1$ ($\because$ vertex is $(\pm 5, 0)$)

$$\Rightarrow 9x^2 - 25y^2 = 225$$

4. (d) Tangent at point $P(a \sec \theta, b \tan \theta)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$$\frac{x \sec \theta}{a} - y \frac{\tan \theta}{b} = 1 \quad (1)$$

Equation of one of asymptote $\Rightarrow \frac{x}{a} + \frac{y}{b} = 0 \quad (2)$

$\Rightarrow$ Coordinates of Q on the asymptote are $(a(\sec \theta - \tan \theta), -b(\sec \theta - \tan \theta))$

Let mid-point of PQ is $M(h, k)$

$$\Rightarrow h = \frac{a \sec \theta + (a \sec \theta - a \tan \theta)}{2}$$

$$\Rightarrow \frac{h}{a} = \sec \theta - \frac{\tan \theta}{2} \quad (3)$$

Similarly, $\frac{k}{b} = \tan \theta - \frac{\sec \theta}{2} \quad (4)$

$$(3) + (4) \Rightarrow \frac{h}{a} + \frac{k}{b} = \frac{\sec \theta + \tan \theta}{2} \quad (5)$$

$$(3) - (4) \Rightarrow \frac{h}{a} - \frac{k}{b} = \frac{3}{2} [\sec \theta - \tan \theta] \quad (6)$$

Now $(5) \times (6) \Rightarrow \frac{h^2}{a^2} - \frac{k^2}{b^2} = \frac{3}{4}$

or, $4\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right) = 3$

5. (b) Let the foot of perpendicular from center upon any normal be $P(h, k)$

Then equation of normal to hyperbola is

$$(y - k) = -\frac{h}{k}(x - h)$$

or $hx + ky = h^2 + k^2 \quad (1)$

Also normal at any point $R(a \sec \theta, b \tan \theta)$ on the hyperbola is

$$\frac{ax}{\sec \theta} + \frac{by}{\tan \theta} = a^2 + b^2 \quad (2)$$

Comparing ratio coefficients of equations (1) and (2),

We get $\frac{a}{h} = \frac{b}{k} = \frac{a^2 + b^2}{h^2 + k^2}$

$$\Rightarrow \sec \theta = \frac{a(h^2 + k^2)}{h(a^2 + b^2)} \text{ and } \tan \theta = \frac{b(h^2 + k^2)}{k(a^2 + b^2)}$$

Squaring and subtracting $(x^2 + y^2)^2 \left(\frac{a^2}{x^2} - \frac{b^2}{y^2} \right) = (a^2 + b^2)^2$

6. (a) Let the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Any tangent to it is

$$\frac{x}{a} \sec \phi - \frac{y}{b} \tan \phi = 1 \quad (1)$$

at $Q(a \sec \phi, b \tan \phi)$

The tangent cuts the axis of y at P .

The coordinates of P are $(0, -b \cot \phi)$

As P is fixed $\Rightarrow b \cot \phi = \lambda$ (say) (2)

Now $x = a \sec \phi, y = b \tan \phi \quad (3)$

Eliminating b and ϕ

from (1), (2), (3), we get $\frac{x^2}{a^2} - \frac{y}{\lambda} = 1$, which clearly is a parabola.

7. (d) Equation of tangent at $P(ct, c/t)$ is $\frac{x}{t} + ty = 2c$

or $x + t^2 y = 2ct \quad (1)$

Slope of tangent $= -\frac{1}{t^2}$

$\therefore$ Equation of perpendicular to this tangent from $(0, 0)$ is $y = t^2 x$

Squaring (i), $(x + t^2 y)^2 = 4c^2 t^2 \quad (2)$

Putting the value of t^2 from (2) in above we get

$$\left(x + \frac{y^2}{x}\right)^2 = 4c^2 \frac{y}{x}$$

$$\Rightarrow (x^2 + y^2) = 4c^2 xy$$

Multiple Correct Answers Type

8. (c, d)

(α, α^2) must lie between the asymptotes of hyperbola

$$\frac{x^2}{1} - \frac{y^2}{4} = 1 \text{ in I and II quadrant. Asymptotes are } y = \pm 2x; \Rightarrow$$

$$2\alpha < \alpha^2 \Rightarrow \alpha < 0 \text{ or } \alpha > 2.$$

$$\text{and } -2\alpha < \alpha^2 \Rightarrow \alpha < -2 \text{ or } \alpha > 0 \Rightarrow \alpha \in (-\infty, -2) \cup (2, \infty)$$

9. (c, d)

Radius of director circle $\sqrt{a^2 - b^2} = 2$ Axis of hyperbola is line joining the center $C(0, 2)$ and $A(2, 0)$ (end of major axis)

$$\therefore a = CA = 2\sqrt{2}$$

$$\therefore (2\sqrt{2})^2 - b^2 = 4$$

$$\therefore b^2 = 4$$

$$\therefore e = \frac{\sqrt{3}}{\sqrt{2}}$$

Center of the hyperbola is center of the director circle $(0, 2)$ Focus lies on this line at distance ae from center

$$\therefore \text{If foci are } (x, y) \text{ then } \frac{x-0}{-\frac{1}{\sqrt{2}}} = \frac{y-2}{\frac{1}{\sqrt{2}}} = \pm 2\sqrt{3}$$

$$(x, y) = (\sqrt{6}, 2 - \sqrt{6}) \text{ or } (-\sqrt{6}, 2 + \sqrt{6})$$

10. (a, b, c)

Required points will lie on the intersection of ellipse

$$\frac{x^2}{2} + \frac{y^2}{10} = 1 \text{ with director circle of hyperbola } \frac{x^2}{5} - \frac{y^2}{1} = 1$$

$$\text{i.e. on } x^2 + y^2 = 4$$

$$\Rightarrow (\sqrt{2} \cos \theta)^2 + (\sqrt{10} \sin \theta)^2 = 4$$

$$\text{Solving, we get } \sin \theta = \pm \frac{1}{2}, \cos \theta = \pm \frac{\sqrt{3}}{2}$$

$$\therefore \text{Points are } \left(\pm \sqrt{\frac{3}{2}}, \pm \sqrt{\frac{5}{2}} \right)$$

Comprehension Type

11. (c), 12. (b), 13. (d)

Let $(2t, 2/t)$ be a point on the hyperbola.Equation of the tangent at this point $x + yt^2 = 4t$.

$$\therefore A = (4t, 0), B = (0, 4/t)$$

Locus of circumcentre of triangle is $xy = 16$ Its eccentricity is $\sqrt{2}$

Shortest distance exist along the common normal.

$$\therefore t^2 = 1/2 \text{ or } t = 1/\sqrt{2}$$

$$\therefore \text{Foot of the perpendicular is } C(\sqrt{2}, 2\sqrt{2})$$

$$\therefore \text{Shortest distance is distance of } C \text{ from the given line which is } \frac{4(\sqrt{2}-1)}{\sqrt{5}}$$
Given line intersect the x -axis at $R(2, 0)$ Any point on this line at distance ' r ' from R is $(2 + r \cos \theta, r \sin \theta)$ If this point lies on hyperbola, then we have $(2 + r \cos \theta)(r \sin \theta) = 4$ Product of roots of above quadratic in ' r ' is $r_1 r_2 = 8/\sin 2\theta$, which has minimum value 8 $\therefore$ Minimum value of $RS \times RT$ is 8

14. (c) The given line must be parallel to asymptotes

$$\Rightarrow \text{Slope of asymptotes are } \frac{3}{2} \text{ and } -\frac{3}{2}$$

$$\Rightarrow \frac{b}{a} = \frac{3}{2} \Rightarrow e = \frac{\sqrt{13}}{2}$$

15. (d) Equation of asymptotes to hyperbola is $\frac{(x-7)^2}{a^2} - \frac{(y+3)^2}{b^2} = 0$

$$\text{or } \frac{b^2(x-7)^2}{a^2} - (y+3)^2 = 0$$

$$\text{or } \frac{9(x-7)^2}{4} - (y+3)^2 = 0$$

Now $\alpha^2 - 4$ is such that two tangents can be drawn to each branch of the given hyperbola, then

$$\frac{9(\alpha+7-7)^2}{4} - (\alpha^2-1)^2 < 0$$

$$\Rightarrow (\alpha^2-1)^2 > \frac{9\alpha^2}{4}$$

$$\Rightarrow (2\alpha^2-3\alpha-2)(2\alpha^2+3\alpha-2) > 0$$

$$\Rightarrow (2\alpha+1)(\alpha-2)(2\alpha-1)(\alpha+2) > 0$$

$$\Rightarrow \alpha \in (-\infty, -2) \cup (-1/2, 1/2) \cup (2, \infty)$$